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Kinetic Theory of Long Time Tails in Velocity Correlation Functions in a Moderately Dense Electron Gas

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The long time tails of the correlation functions that determine the self-diffusion coefficient and the kinetic parts of the shear viscosity and heat conductivity in a one-component plasma are calculated using a systematic kinetic theory. The results are in agreement with those obtained from the phenomenological mode coupling theory. The formal kinetic theory calculations of previous workers, who obtained incomplete long time tail results, are also discussed.

KEY WORDS: Electron gas; time correlation functions; long time tails; kinetic theory.

1. INTRODUCTION

In the past decade there has been considerable interest in the time dependence of the velocity correlation functions that determine the transport coefficients in a dense electron gas or one-component plasma (OCP). These time correlation functions have been the object of both extensive theoretical work[1-3] and computer simulations.[4,5] A large part of the theoretical work has been based on formal kinetic theories that are in general not systematic.[1,2] Nevertheless, these theories have been successful in fitting the results of computer experiments even in very dense systems.[6] Their prediction for the asymptotic long-time behavior of the time correlation functions for the transport coefficients differs, however, from that of a phenomenological hydrodynamic mode coupling theory.

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Phenomenological mode coupling theories are well tested for neutral fluids, where they are in complete agreement with microscopic calculations.[7] The question then naturally arises whether the mode coupling theory breaks down for systems where the particles interact through long range forces or whether the formal kinetic theories used in the literature are in error. In this paper we address this question. We analyze the long time behavior of the velocity autocorrelation function and of the kinetic part of the Green–Kubo integrands for the shear viscosity and the heat conductivity using the kinetic theory developed in detail elsewhere.[8] The theory is systematic since it is based on a detailed analysis of the density expansion of the velocity correlation functions and the selection, in each order in the density, of those classes of collision sequences that are most important in the limit of interest. We find that the formal kinetic theories are incomplete and in general fail to identify the asymptotic long time behavior. The results of our work are in complete agreement with those of the mode coupling theory.[9,10]

It is important to stress that here we analyze only the kinetic parts of the shear current and heat current autocorrelation functions. For neutral fluids consideration of the kinetic part is sufficient to identify all the mode coupling mechanisms that contribute to the leading long-time behavior of the full Green–Kubo integrands. The analysis of the potential contributions to the shear viscosity and heat conductivity leads to no qualitatively new features—it only provides density corrections to the coefficients of the long time tails.[11]

The situation is quite different for an electron gas. The potential part of the correlation functions contains contributions to the leading long time tails that are of the same order in the plasma parameter as those arising from the kinetic part and may even cancel some of the kinetic contributions (as happens for the shear current autocorrelation function). This point will be discussed in more detail below. Complete results for the long time tails of both the kinetic and potential parts of the shear current, heat current, and longitudinal current autocorrelation functions have been obtained using a phenomenological mode coupling theory and have been given elsewhere.[10]

Gould and Mazenko[1] and Baus and Wallenborn[2] were the first to use the techniques of kinetic theory to calculate the long-time behavior of the time correlation functions that determine the transport coefficients of an OCP. Gould and Mazenko[1] examined the velocity autocorrelation function (VACF) for a tagged particle in a moderately dense OCP. They found that this function has a slowly decaying oscillating long-time tail proportional to

$$t^{-3/2} \sin \omega_p t \tag{1.1}$$

Here t is the time and $\omega_p = [4\pi e^2 n/m]^{1/2}$ is the plasma frequency, where n is the number density of the moving particles, m is their mass and e their charge. This oscillating long-time tail is to be contrasted with the $t^{-3/2}$ long-time decay of the VACF in neutral fluids.[11] Furthermore, recent mode coupling theory calculations for the OCP have shown the VACF has an additional contribution that is proportional to $t^{-3/2}$.[9,10] This algebraic long-time decay results from the coupling between a self-diffusion hydrodynamic mode and a viscous or shear hydrodynamic mode. The kinetic theory of Gould and Mazenko predicts that the amplitude for such coupling vanishes and the velocity autocorrelation function has no purely decaying $t^{-3/2}$ long-time tail.

Baus and Wallenborn[2] have used a formal kinetic theory to calculate the long-time behavior of the time correlation function that determines the shear viscosity. They considered both kinetic and potential parts and showed that the slow long time decay of such a correlation function is due to the coupling of hydrodynamic modes. In their theory the only important mechanism of mode coupling is, however, that of two plasma modes. They found that the leading asymptotic decay of the shear current autocorrelation function is proportional to

$$(\Gamma t)^{-3/2}[1 + A\cos(2\omega_p t + \phi)] \tag{1.2}$$

Here Γ is the damping of the plasma modes and A and ϕ are constants. A simple mode coupling theory predicts, however, that the coupling of two plasma modes can only lead to an oscillating long time tail $\sim t^{3/2} \cos(2\omega_p t + \phi)$, while a purely decaying contribution, $\sim t^{-3/2}$, is obtained from the coupling between two transverse shear modes. Such a coupling mechanism is not present in the kinetic theory of Baus and Wallenborn, since the corresponding amplitude vanishes in their theory.

In this paper we extend the low-density kinetic theory presented in Ref. 8, to be denoted in the following as I, to describe a moderately dense OCP. This is done by a systematic analysis of the collision diagrams that appear in the density expansion of the velocity correlation functions. We show that the systematic theory leads to long-time tails (LTT) in the velocity correlation functions in an OCP that are in agreement with those computed on the basis of mode coupling theory.[3] The collision sequences that generate these long-time tails are identical in structure to the collision sequences that are responsible for the long-time tails in a neutral fluid.[11] This will be discussed in greater detail below. In the course of the calculations we also discuss the previous kinetic theory calculations.

[3] Using the hydrodynamic mode coupling theory of Kadanoff and Swift[12] it is possible to evaluate separately the kinetic and potential parts of the correlation functions.

The plan of this paper is as follows. In Section 2 we review the results of I and discuss the qualitative structure of our results, comparing them to the previous kinetic theories. In Section 3 we present a systematic derivation of the ring kinetic equation in a OCP that leads to long time tails which are in agreement with the mode coupling theory. In Section 4 we present our results and explicitly calculate the long time behavior of the VACF and the velocity correlation functions that determine the kinetic parts of the shear viscosity, η, and the heat conductivity, λ. In Section 5 we conclude this paper with a discussion.

2. BASIC EQUATIONS AND STRUCTURE OF THE THEORY

2.1. Review of I

To simplify the presentation, only the VACF is explicitly considered here. The results for the shear viscosity and the thermal conductivity will be quoted in Section 4.

The VACF is defined as

$$\begin{aligned} C_D(t) &= \lim_{\substack{N,\Omega \to \infty \\ N/\Omega = n}} \langle v_{1x}(t) v_{1x} \rangle \\ &= \lim_{\substack{N,\Omega \to \infty \\ N/\Omega = n}} \int dx_N \, v_{1x} S_{-t}(x^N) \, \rho_N(x^N) v_{1x} \\ &= \int d\mathbf{v}_1 \, v_{1x} \phi(v_1) \, \Phi_D(\mathbf{v}_1, t) \end{aligned} \tag{2.1a}$$

where N is the number of particles, Ω is the volume of the system, $x_i = (\mathbf{r}_i, \mathbf{v}_i)$ denotes the position and velocity of the ith particle, $S_{-t}(x^N) = \exp[-tL_N]$ is the N-particle streaming operator, and L_N is the N-particle Liouville operator, given by

$$L_N = \sum_{i=1}^{N} L_0(x_i) - \frac{1}{2} \sum_{i \neq j}^{N} \theta_{ij} \tag{2.1b}$$

with $L_0(x_i)$ the Liouville operator for a single particle i,

$$L_0(x_i) = \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{r}_i} \tag{2.1c}$$

and,

$$\theta_{ij} = \frac{1}{m} \frac{\partial V(r_{ij})}{\partial \mathbf{r}_{ij}} \cdot \left(\frac{\partial}{\partial \mathbf{v}_i} - \frac{\partial}{\partial \mathbf{v}_j} \right) \tag{2.1d}$$

Here $r_{ij} = |r_i - r_j|$ and $V(r_{ij})$ is the Coulomb potential between particles i and j, given by

$$V(r_{ij}) = e^2/|\mathbf{r}_i - \mathbf{r}_j| \tag{2.1e}$$

In Eq. (2.1) $\rho_N(x^N)$ is the equilibrium N-particle distribution function in the canonical ensemble and $\phi(v)$ is the Maxwell distribution function,

$$\phi(v) = \left(\frac{\beta m}{2\pi}\right)^{3/2} \exp[-\beta m v^2/2] \tag{2.1f}$$

By using standard cluster expansion techniques[8,11] a density expansion for the time derivative of $\Phi_D(\mathbf{v}_1, t)$ can be obtained, with the result

$$\frac{\partial \Phi_D(\mathbf{v}_1, t)}{\partial t} = \sum_{s=2}^{\infty} n^{s-1} \mathcal{A}_s(\mathbf{v}_1, t) v_{1x} \tag{2.2}$$

where $\mathcal{A}_s(\mathbf{v}_1, t)$ is an s-particle operator that contains both dynamical operators and equilibrium distribution functions. The first few operators $\mathcal{A}_s$ are given by

$$\mathcal{A}_2(\mathbf{v}_1, t) = \phi^{-1}(v_1) \int d2\theta_{12} U(12, t)\, g_2^{(0)}(\mathbf{r}_1, \mathbf{r}_2)\, \phi(v_2)\, \phi(v_1) \tag{2.3a}$$

and

$$\mathcal{A}_1(\mathbf{v}_1, t) = \phi^{-1}(v_1) \int d2d3\theta_{12} U(12|3, t)\, g_3^{(0)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\, \phi(v_3)\, \phi(v_2)\, \phi(v_1)$$

$$+ \phi^{-1}(v_1) \int d2\theta_{12} U(12, t)\, g_2^{(1)}(\mathbf{r}, \mathbf{r}_2)\, \phi(v_2)\, \phi(v_1) \tag{2.3b}$$

and

$$\mathcal{A}_4(\mathbf{v}_1, t) = \frac{\phi^{-1}(v_1)}{2!} \int d2\theta_{12} \left[\int d3d4 U(12|34, t)\, g_4^{(0)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)\, \phi(v_4)\, \phi(v_3) \right.$$

$$+ 2 \int d3 U(12|3, t)\, g_3^{(1)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\, \phi(v_3)$$

$$\left. + 2U(12, t)\, g_2^{(2)}(\mathbf{r}_1, \mathbf{r}_2) \right] \phi(v_2)\, \phi(v_1) \tag{2.3c}$$

etc., where $l = x_l = (\mathbf{r}_l, \mathbf{v}_l)$. Here $U(12|3, 4,..., s, t)$, for $s = 3, 4,..., N$, are s-particle dynamical cluster operators, given by

$$U(12, t) = S_{-t}(12) \tag{2.4a}$$

and,

$$U(12|3, t) = S_{-t}(123) - S_{-t}(12)\, S_{-t}(3) \tag{2.4b}$$

etc.

In Eq. (2.3) we have introduced the density expansion of the equilibrium configurational space distribution functions, g_s. The latter are defined by

$$f_s(x_1,..., x_s) = n^s g_s(\mathbf{r}_1, \mathbf{r}_2,..., \mathbf{r}_s)\, \phi(v_1) \cdots \phi(v_s) \tag{2.5a}$$

where f_s is the s-article phase space reduced distribution function in equilibrium, defined in Eq. (A.1). The virial expansion of g_s has the form

$$g_s(\mathbf{r}_1, \mathbf{r}_2,..., \mathbf{r}_s) = \sum_{l=0}^{\infty} n^l g_s^{(l)}(\mathbf{r}_1, \mathbf{r}_2,..., \mathbf{r}_s) \tag{2.5b}$$

The properties of $g_s^{(l)}$ used in this paper can be found for instance in Ref. 13. Similarly, we will write

$$f_s(x_1,..., x_s) = \sum_{\rho=0}^{\infty} n^l f_s^{(l)}(x_1,..., x_s) \tag{2.5c}$$

with

$$f_s^{(l)}(x_1,..., x_s) = \phi(v_1) \cdots \phi(v_s)\, g_s^{(l)}(\mathbf{r}_{1,...,}\, \mathbf{r}_s) \tag{2.5b}$$

To proceed, it is convenient to introduce the Laplace transform of Φ_D, defined as

$$\Phi_D(\mathbf{v}_1, z) = \int_0^{\infty} dt\, e^{-zt} \Phi_D(\mathbf{v}_1, t) \tag{2.6}$$

for Re$z > 0$. Using $\Phi_D(\mathbf{v}_1, t=0) = v_{1x}$, Eq. (2.2a) yields

$$z\Phi_D(\mathbf{v}_1, z) = \left[1 + \sum_{s=2}^{\infty} n^{s-1} \mathcal{O}\!\mathcal{C}_s(\mathbf{v}_1, z) \right] v_{1x} \tag{2.7}$$

where $\mathcal{O}\!\mathcal{C}_s(v_1, z)$ is the Laplace transform of $\mathcal{O}\!\mathcal{C}_s(\mathbf{v}_1, t)$. Equation (2.7) contains divergences in the limit Re$z \to 0$ due to the contribution to the dynamical operators $\mathcal{O}\!\mathcal{C}_s$ from sequences of $s-1$ collisions among s particles. A kinetic equation for $\Phi_D(\mathbf{v}_1, z)$ free of such naive divergences can be obtained by using the inversion procedure developed for neutral fluids.[14] The resulting equation is

$$\left[z - \sum_{l=2}^{\infty} n^{l-1} B_l(\mathbf{v}_1, z) \right] \Phi_D(\mathbf{v}_1, z) = v_{1x} \tag{2.8a}$$

where $B_l(\mathbf{v}_1, z)$ is an l-particle collision operator. The first few operators B_l are

$$B_2(\mathbf{v}_1, z) = z\mathscr{A}_2(\mathbf{v}_1, z) \tag{2.8b}$$

and

$$B_3(\mathbf{v}_1, z) = z\mathscr{A}_3(\mathbf{v}_1, z) - z[\mathscr{A}_2(\mathbf{v}_1, z)]^2 \tag{2.8c}$$

and

$$B_4(\mathbf{v}_1, z) = z\mathscr{A}_4(\mathbf{v}_1, z) - z\mathscr{A}_2(\mathbf{v}_1, z)\,\mathscr{A}_3(\mathbf{v}_1, z) - z\mathscr{A}_3(\mathbf{v}_1, z)\,\mathscr{A}_2(\mathbf{v}_1, z) + z[\mathscr{A}_2(\mathbf{v}_1, z)]^3 \tag{2.8d}$$

etc.

In I the kinetic equation given by Eqs. (2.8) was analyzed for a dilute electron gas characterized by a small value of the plasma parameter, $\varepsilon_p = (4\pi n\lambda_D^3)^{-1}$, where λ_D is the Debye screening length, $\lambda_0 = (4\pi\beta ne^2)^{-1/2}$. We found there that each term in the density expansion of the collision operator in Eq. (2.2a) diverges. By resumming the most divergent terms to every order in the density we obtained a non-Markoffian Balescu–Guernsey–Lenard (BGL) equation for $\Phi_D(\mathbf{v}_1, z)$, given by

$$[z - \Lambda_s^0(\mathbf{v}_1, z)]\,\Phi_D(\mathbf{v}_1, z) = v_{1x} \tag{2.9a}$$

Here $\Lambda_s^0(\mathbf{v}_1, z)$ is the homogeneous "self" BGL collision operator,

$$\Lambda_s^0(\mathbf{v}_1, z) = n\phi^{-1}(v_1)\int d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3}\,\theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\,\frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{12} - \mathscr{V}_{\mathbf{q}}(\mathbf{v}_2)} \times \phi(v_1)\,\phi(v_2)\,\hat{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \tag{2.9b}$$

In Eq. (2.9b), $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$ and $\theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)$ is the Fourier transform of θ_{12},

$$\theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) = \frac{\boldsymbol{\varepsilon}_{\mathbf{q}}}{m}\cdot\left(\frac{\partial}{\partial\mathbf{v}_1} - \frac{\partial}{\partial\mathbf{v}_2}\right) \tag{2.10a}$$

with $\boldsymbol{\varepsilon}_{\mathbf{q}} = i\mathbf{q}V_{\mathbf{q}}$, where $V_{\mathbf{q}} = 4\pi e^2/q^2$ is the Fourier transform of the Couloumb potential. Also, $\mathscr{V}_{\mathbf{q}}(\mathbf{v}_2)$ is the Vlasov operator for particle two,

$$\mathscr{V}_{\mathbf{q}}(\mathbf{v}_2) = \frac{n}{m}\int d\mathbf{v}_3\,\boldsymbol{\varepsilon}_{\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{v}_2}\,\phi(v_2)\,P_{23} \tag{2.10b}$$

with P_{23} a permutation operator that permutes the labels 2 and 3. A Vlasov operator for particle one does not appear in Eq. (2.9b) since self-diffusion is considered here. In Eq. (2.9b), $\hat{\theta}$ is given by

$$\hat{\theta}_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) = \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) + \int d\mathbf{v}_3 \, nh_{\mathrm{DH}}(q) \, \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_3) \, \phi(v_3)$$

$$= S_{\mathrm{DH}}(q) \frac{1}{m} \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} - \frac{1}{m} \boldsymbol{\varepsilon}_{\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_2} \tag{2.10c}$$

where $h_{\mathrm{DH}}(q)$ and $S_{\mathrm{DH}}(q)$ are, respectively, the pair correlation function and the static structure factor in the Debye–Hückel approximation, given by

$$S_{\mathrm{DH}}(q) = 1 + nh_{\mathrm{DH}}(q) = \frac{(q\lambda_{\mathrm{D}})^2}{1 + (q\lambda_{\mathrm{D}})^2} \tag{2.10d}$$

In the remainder of the paper we will derive corrections to the BGL kinetic equation, (2.9). To do this, it is convenient to use a diagrammatic technique. The rules adopted to construct the diagrams are as follows.

(1) A diagram consists of vertical lines, horizontal bonds, crosses, and thick black circles. The vertical lines are labeled at their bottom with the particle labels 1, 2,.... The velocities of all particles except 1 are integrated over.

(2) The operators and functions corresponding to the elements of a diagram are multiplied in the same order from left to right as they appear from top to bottom in the diagram. The same diagrammatic representation is used for collision sequences in time language and in Laplace language.

(3) A cross and a vertical line segment at the top, ⊤̄, or bottom, ⊥̄, level of a diagram denote root points. For the VACF both root points have the particle label 1.

(4) There is a factor $n\phi(v_i)$ associated with each label i that is not a root point. The location of $n\phi(v_i)$ in a diagram will be denoted by a thick dark circle, ●. Each diagram must also be multiplied by $\phi^{-1}(v_1)$.

(5) Horizontal bonds are of three types: (i) statistical bonds that represent factors of the potential, as obtained from the potential expansion of the Mayer f functions, and are drawn as dashed lines: $-\beta V_{\mathbf{q}} =$ •- - - - -• ; (ii) dynamical bonds or θ bonds representing the two-particle operator θ_{ij} and drawn as wavy lines, $\theta_{\mathbf{q}}(\mathbf{v}_i, \mathbf{v}_j)$ •~~~~~~• ; (iii) $\bar{\theta}$ bonds, representing the *function* $\bar{\theta}_{\mathbf{q}}(\mathbf{v}_i, \mathbf{v}_j) = -[\phi(v_i)\,\phi(v_j)]^{-1}$ $[\theta_{\mathbf{q}}(\mathbf{v}_i, \mathbf{v}_j)\,\phi(v_i)\,\phi(v_j)]$ and drawn as dotted lines, i.e., $\bar{\theta}_{\mathbf{q}}(\mathbf{v}_i, \mathbf{v}_j) =$ •·······•. The $\bar{\theta}$ bonds arise because the kinetic energy is only conserved in the long time limit in a binary collision.

(6) Momentum or wave vector is conserved at each vertex and each internal wave vector is to be integrated over with a factor $(2\pi)^{-3}$.

(7) The vertical position of θ bonds and $\bar{\theta}$ bonds defines the levels in the diagram. In time language the bottom level corresponds to the smallest time and an ordered time integration, $\int_0^t dt_1 \int_{t_1}^t dt_2 \int_{t_2}^t dt_3, \ldots$, is performed over the times of all intermediate levels.

(8) Internal vertical line segments between diagram levels represent free propagation of the particles, i.e., $\mathbf{q}\updownarrow_0^t = \exp(-i\mathbf{q}_j \cdot \mathbf{v}_j t)$.

(9) A double vertical line segment between diagram levels denotes a free propagator that has been renormalized by a Vlasov operator, i.e.,

$$\mathbf{q}\Updownarrow_0^t = \exp\{-t[i\mathbf{q} \cdot \mathbf{v}_j - \mathscr{V}_{\mathbf{q}}(\mathbf{v}_j)]\}$$

(10) Thick dashed lines denotes static bonds that have been renormalized to a Debye–Hückel pair correlation function, i.e.,

$$h_{\mathrm{DH}}(q) = {}_i\searrow\underset{q}{-}\nearrow_j = -\frac{1}{n}(1 + q^2 \lambda_D^2)^{-1}$$

With these diagram rules the diagrammatic representation of $\Lambda_s^0(\mathbf{v}_1, z)$ is given in Fig. 1a. Here $\Lambda_s^0(\mathbf{v}_1, z)$ has been represented by the sum of two diagrams, corresponding to the two terms obtained by inserting Eq. (2.10c) into Eq. (2.9b). In Fig. 1b we show the static diagrams that determine the Debye–Hückel pair correlation function, $h_{\mathrm{DH}}(r_{12})$, and in Fig. 1c we show the diagrams that are summed to obtain the renormalized Vlasov propagator, $\mathscr{V}(2)$.

Our objective here is to derive the corrections to Eq. (2.9) that will enable us to consistently calculate the long time tail of the VACF in a moderately dense OCP, where $\varepsilon_p < 1$. In the range $\varepsilon_p < 1$ we can consistently solve Eq. (2.8a) by expanding about the solution of the BGL equation, Eq. (2.9).

Equation (2.8a) is rewritten as

$$[z - \Lambda_s^0(\mathbf{v}_1, z) - \Delta B(\mathbf{v}_1, z)]\, \Phi_D(\mathbf{v}_1, z) = v_{1x} \qquad (2.11a)$$

where

$$\Delta B(\mathbf{v}_1, z) = \sum_{l=2}^{\infty} n^{l-1} B_l(\mathbf{v}_1, z) - \Lambda_s^0(\mathbf{v}_1, z) \qquad (2.11b)$$

Equation (2.11a) can be solved iteratively by writing

$$\Phi_D(\mathbf{v}_1, z) = \Phi_D^{(0)}(\mathbf{v}_1, z) + \Phi_D^{(1)}(\mathbf{v}_1, z) + \cdots \qquad (2.12a)$$

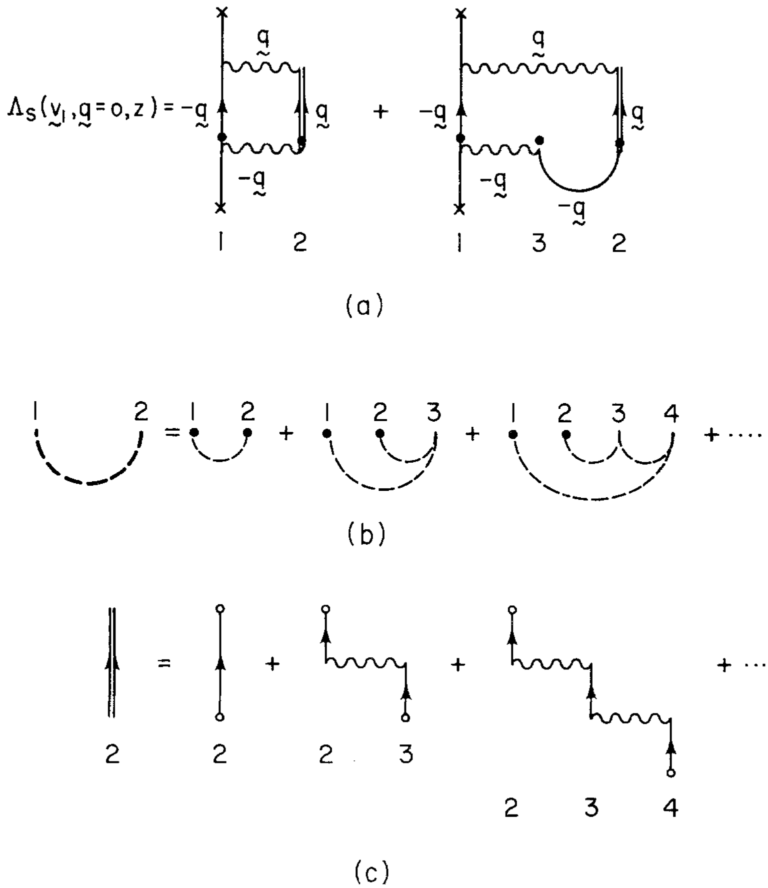

Fig. 1. (a) Diagrammatic representation of $\Lambda_s^0(\mathbf{v}_1, z)$, given by Eq. (2.9b); (b) first few diagrams that contribute to the Debye–Hückel pair correlation function, $h_{\rm DH}(r_{12})$; (c) first few diagram that determine the renormalized Vlasov propagator, (2).

where $\Phi_D^{(0)}(\mathbf{v}_1, z)$ satisfies the BGL equation whose formal solution is

$$\Phi_D^{(0)}(\mathbf{v}_1, z) = [z - \Lambda_s^0(\mathbf{v}_1, z)]^{-1} v_{1x} \tag{2.12b}$$

$\Phi_D^{(1)}(\mathbf{v}_1, z)$ is immediately evaluated by using Eqs. (2.10) and (2.11). It is given by

$$\Phi_D^{(1)}(\mathbf{v}_1, z) = [z - \Lambda_s^0(\mathbf{v}_1, z)]^{-1} \Delta B(\mathbf{v}_1, z)[z - \Lambda_s^0(\mathbf{v}_1, z)]^{-1} v_{1x} \tag{2.12c}$$

The ellipsis in Eq. (2.12a) represent higher-ord$_r$ iterates, leading to plasma parameter corrections to $\Phi_D^{(1)}$.

2.2. Qualitative Structure of the Theory

In this section we discuss the qualitative results of our theory for the long time tails of the VACF for a moderately dense OCP.

Physically one anticipates that the long time tails in the VACF are due to correlated collision sequences where, for example, particles 1 and 2 interact, then propagate via hydrodynamic modes and then interact again. It is this type of mechanism that causes the long time tails in neutral fluids. One might then guess, that the long time tails in a OCP could be obtained from a collision operator like that given in Eq. (2.9b) where the intermediate two-particle propagator contains, however, hydrodynamic modes. If we denote such an approximation for the kinetic operator ΔB by $\Delta B_{\theta\theta}$, then we would expect

$$\Delta B_{\theta\theta}(\mathbf{v}_1, z) = n\phi^{-1}(v_1) \int dv_2 \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\, G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) \times \phi(v_1)\,\phi(v_2)\,\hat{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) - \Lambda_s^0(\mathbf{v}_1, z) \tag{2.13a}$$

where $G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$ is the renormalized two-particle propagator,

$$\begin{aligned} G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) &= \int_0^\infty dt\, e^{-zt} \Gamma^{(s)}_{-\mathbf{q}}(\mathbf{v}_1, t)\, \Gamma_{\mathbf{q}}(\mathbf{v}_2, t) \\ &= \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dz_1}{2\pi i} [z_1 - i\mathbf{q}\cdot\mathbf{v}_1 - \tilde{\Lambda}_s(\mathbf{v}_1, -\mathbf{q}, z_1]^{-1} \\ &\quad \times [z - z_1 + i\mathbf{q}\cdot\mathbf{v}_2 - \mathscr{V}_{\mathbf{q}}(\mathbf{v}_2) - \tilde{\Lambda}(\mathbf{v}_2, \mathbf{q}, z - z_1)]^{-1} \end{aligned} \tag{2.13b}$$

Here γ defines a contour to the right of all the singularities of $(z_1 - i\mathbf{q}\cdot\mathbf{v}_1 - \tilde{\Lambda}_s(\mathbf{v}_1, -\mathbf{q}, z_1)]^{-1}$ and $\Gamma(\Gamma^{(s)})$ is the BGL propagator for fluid (tagged) particle motion in time representation. In Eq. (2.13b) $\tilde{\Lambda}_s(\mathbf{v}_1, \mathbf{q}, z)$ is the inhomogenous BGL collision operator for tagged particle collisions, given by

$$\begin{aligned} \tilde{\Lambda}_s(\mathbf{v}_1, -\mathbf{q}, z)\,\phi(v_1) &= \phi(v_1)\,\Lambda_s(\mathbf{v}_1, -\mathbf{q}, z) \\ &= n \int d\mathbf{v}_2 \int \frac{d\mathbf{q}'}{(2\pi)^3} \theta_{\mathbf{q}'}(\mathbf{v}_1, \mathbf{v}_2)[z + i(\mathbf{q} - \mathbf{q}')\cdot\mathbf{v}_1 + i\mathbf{q}'\cdot\mathbf{v}_2 - \mathscr{V}_{\mathbf{q}'}(\mathbf{v}_2)]^{-1} \\ &\quad \times \phi(v_1)\,\phi(v_2)\,\hat{\theta}_{-\mathbf{q}'}(\mathbf{v}_1, \mathbf{v}_2) \end{aligned} \tag{2.14a}$$

It is related to the homogeneous operator defined in Eq. (2.9b) by

$$\Lambda_s^0(\mathbf{v}_1, z) = \Lambda_s(\mathbf{v}_1, \mathbf{q} = 0, z) \tag{2.14b}$$

and

$$\tilde{\Lambda}_s^0(\mathbf{v}_1, z) = \tilde{\Lambda}_s(\mathbf{v}_1, \mathbf{q} = 0. z) \tag{2.14c}$$

Finally, $\tilde{\Lambda}(\mathbf{v}_2, \mathbf{q}, z)$ is the inhomogenous collision operator for fluid particle collisions,

$$\begin{aligned}\tilde{\Lambda}_s(\mathbf{v}_2, \mathbf{q}, z)\,\phi(v_2) &= \phi(v_2)\,\Lambda(\mathbf{v}_2, \mathbf{q}, z)\\ &= n \int d\mathbf{v}_3 \int \frac{d\mathbf{q}'}{(2\pi)^3}\,\theta_{\mathbf{q}'}(\mathbf{v}_2, \mathbf{v}_3)[z + i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{v}_2 + i\mathbf{q}'\cdot\mathbf{v}_3\\ &\quad - \mathscr{V}_{\mathbf{q}-\mathbf{q}'}(\mathbf{v}_2) - \mathscr{V}_{\mathbf{q}'}(v_3)]^{-1}\,\phi(v_2)\,\phi(v_3)\\ &\quad \times [\hat{\theta}_{-\mathbf{q}'}(\mathbf{v}_2, \mathbf{v}_3) + \hat{\theta}_{\mathbf{q}-\mathbf{q}'}(v_3, v_2) P_{12}]\end{aligned} \tag{2.1d}$$

It should be remarked that throughout the calculations outlined in this paper one must be very careful about the location of the Maxwell distribution functions since the kinetic energy is not conserved in a binary collision. The appearance of the $\bar{\theta}$ bonds in diagram rule (5) is a consequence of this. However, as discussed in I, for long times or small z our collision operators do conserve kinetic energy so that in this limit $\tilde{\Lambda}_s(\mathbf{v}_1, \mathbf{q}, z \to 0) = \Lambda_s(\mathbf{v}_1, \mathbf{q}, z \to 0)$, etc.

In Fig. 2a we illustrate the diagrammatic representation of the first term in Eq. (2.13). The thick lines denote that the intermediate propagators contain the effects of collisional damping and hydrodynamic modes (cf. Section 4). In Figs. 2b and 2c we show the first few diagrams that determine the renormalized propagators of Fig. 2a.[4]

When the propagators in Eq. (2.13b) are decomposed on the basis of the hydrodynamic modes (cf. Section 4) the LTT of $\Phi_D(\mathbf{v}_1, t)$ given by Eq. (1.1) is obtained because of the coupling of the self-diffusion mode with the plasma modes.[(1)] This result is to be contrasted with that of the phenomenological mode coupling theory, which in addition leads to a $t^{-3/2}$ tail, due to the coupling between the self-diffusion hydrodynamic mode and a shear hydrodynamic mode. This coupling mechanism is not contained in Eq. (2.13) (the corresponding amplitude vanishes) and hence only an oscillating LTT result from the analysis of $\Delta B_{\theta\theta}$.

With this in mind it is worthwhile to reconsider the structure of Eq. (2.13) and compare it with the ring operator that leads to the LTT in neutral fluids. Although they are very similar there is one important physical difference. In Eq. (2.13) the first and last collisions are weak

[4] We have used the same diagrammatic notation for the tagged and fluid particle BGL propagator. Both are represented by thick vertical line segments.

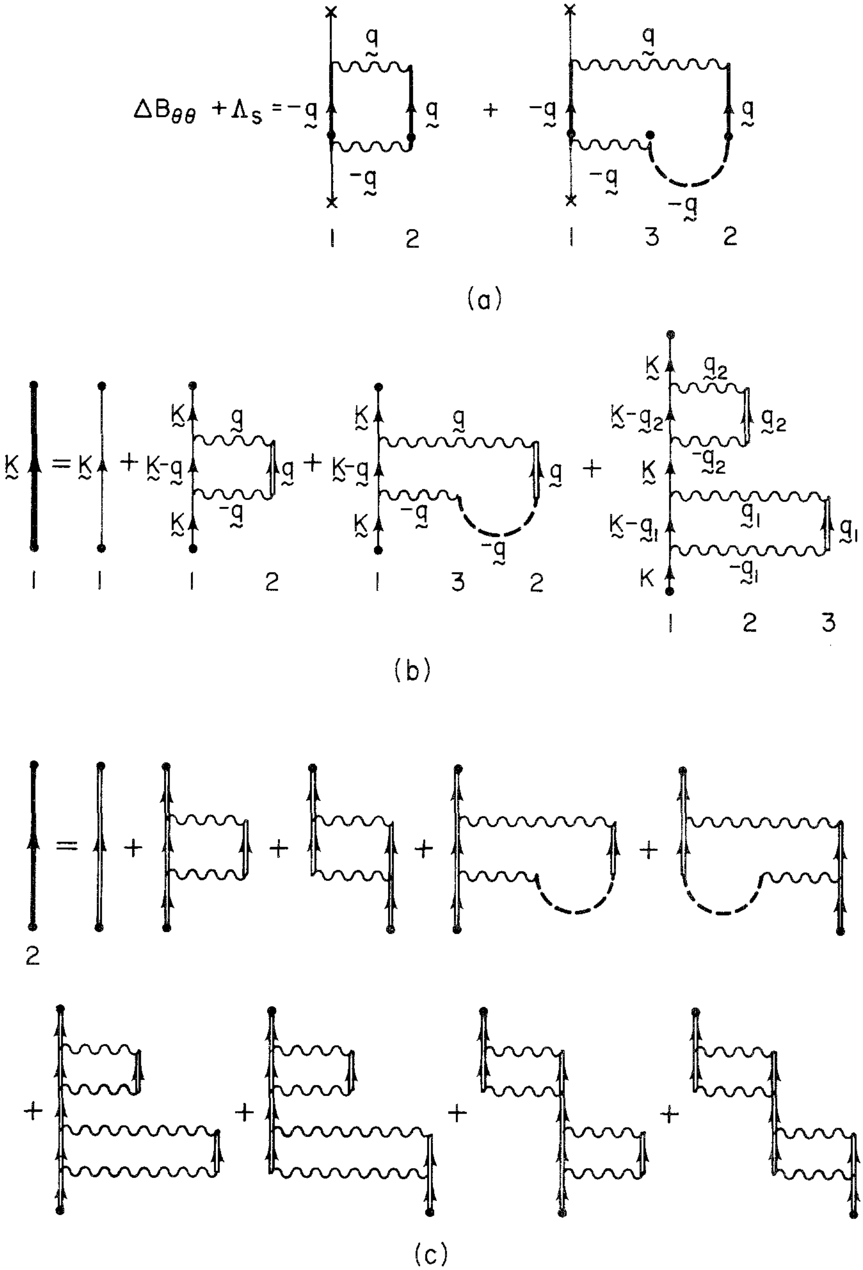

Fig. 2. (a) Diagrammatic representation of Eq. (2.13); (b) first few diagrams that determine the hydrodynamic tagged particle propagator; (c) first few diagrams that determine the hydrodynamic fluid particle propagator.

collisions, involving only one θ operator, whereas in the usual ring operator that leads to the LTT the first and last collisions are strong collisions, involving at least two θ operators. A collision is defined here as strong when the velocities of the colliding particles are correlated at the end of the collision, i.e., the action of the binary collision operator for particles 1 and 2 on a function of $\mathbf{v}_1$ results in a function not only of $\mathbf{v}_1$, but also of the relative velocity, $\mathbf{v}_{12}$. This is, for example, the case for a hard sphere's binary collision. In contrast, the action of θ_{12} on a function of $\mathbf{v}_1$ always leads to a function of $\mathbf{v}_1$ only. The interaction governed by the operator θ_{ij} does not result in velocity correlations and will be referred to as a weak collision. A strong collision is the process governed by the kernel of the BGL operator, which can be regarded as an effective binary collision operator in a OCP. This is an important point since physically one expects that the correlations that lead to the LT are most effectively established by strong collisions. Indeed we show below that when the density expansion of the collision operator ΔB defined in Eq. (2.11) is analyzed systematically, one obtains, in addition to Eq. (2.13), a collision operator that has the same structure as $\Delta B_{\theta\theta}$, where the first and last collisions are, however, strong collisions. We then show that this new strong collision ring operator leads to the same LTT as the phenomenological mode coupling theory. Furthermore, a careful analysis of the diagrammatic expansion also leads to the identification of operators that are similar to Eq. (2.13), except that only the first *or* last collisions are strong collisions. We show that these operators do not lead to a LTT proportional to $t^{-3/2}$.

Finally, it is easy to show that weak collisions involving only one θ operator in fluids interacting through weak short-ranged potentials do *not* lead to the usual long time tails in these fluids.

3. DERIVATION OF THE STRONG COLLISION RING OPERATOR

In this section we derive a ring collision operator for the OCP where both the first and last collisions are strong collisions. The basic idea—which makes the theory systematic—is to first sum all the graphs that renormalize the propagators between any initial and final collision to hydrodynamic propagators. This defines right and left vertex functions. The vertex functions can be again expanded in powers of the density for small wave number of the intermediate hydrodynamic propagators. They can then be systematically evaluated for a dilute—or moderately dense—OCP by retaining only those contributions which are most divergent to every order in the density or plasma parameter. The result is that the vertex functions become BGL-like collision kernels which incorporate the effects of strong initial and final collisions. In Section 4 we show that this ring operator leads to LTT which are in complete agreement with the results of mode coupling theory. In the present work we assume that the two-mode-coupling terms are responsible for the dominant LTT and that the contributions involving the coupling of three or more modes can be neglected. An explicit calculation that justifies this for neutral fluids has been given elsewhere.[15]

The derivation of the ring operator for OCP is considerably more complicated than the corresponding derivation for a neutral fluid. This difficulty stems from the long range nature of the Couloumb potential. Infinite resummations are needed to properly take into account the dynamical screening in constructing the BGL-like collision kernels. Further, as already discussed in I, static correlations must also be treated carefully even for a dilute OCP, due again to the long range nature of the potential.

Before presenting the details of the derivation, it is instructive to summarize the results. By analyzing the operator ΔB for a moderately dense OCP, we identify four kinetic operators that have the structure of ring operators. In all these operators the intermediate propagator is the renormalized two particle propagator G_s, defined in Eq. (2.13b), that contains hydrodynamic modes. The four ring-like operators differ according to whether the first and/or last collisions are weak or strong collisions. The operator where both the first and last collisions are weak was defined in the previous section and denoted by $\Delta B_{\theta\theta}$. Here we will refer to this contribution as $\theta G_s \theta$. The other three kinetic operators have the structure $T_s^L G_s \theta$, $\theta G_s T_s^R$, and $T_s^L G_s T_s^R$, where T_s^R (T_s^L) denotes the operator or vertex corresponding to a strong effective binary collision to the right (left). It is shown below that T_s^R (T_s^L) has a structure analogous to that of the kernel of the BGL operator Λ_s [$\Lambda_s^\dagger$, where the dagger denotes the adjoint, as defined in Eq. (B1)]. As previously mentioned, only the $T_s^L G_s T_s^R$ term contributes to the leading power law long time decay of the VACF.

The general two-mode diagram containing all the four contributions listed above is shown in Fig. 3a. Here the hatched boxes denote complete left (top) and right (bottom) vertex functions, containing both the weak collision operator, θ, the and strong collision operators, T_s^L and T_s^R, respectively. In Figs. 3b and 3c the complete top and bottom vertex functions are written explicitly as the sum of terms corresponding to weak collisions, represented by θ bonds, and terms corresponding to strong collisions, i.e., to the binary operators T_s^L and T_s^R, represented here as shaded boxes. In the diagrammatic represention of the right vertex function we have omitted the diagrams containing $\bar{\theta}$ bonds: the complete right weak collision operator is given by the first two diagrams on the right-hand side of Fig. 3c plus the same two diagrams with θ-bond replaced by a $\bar{\theta}$ bond. By the last

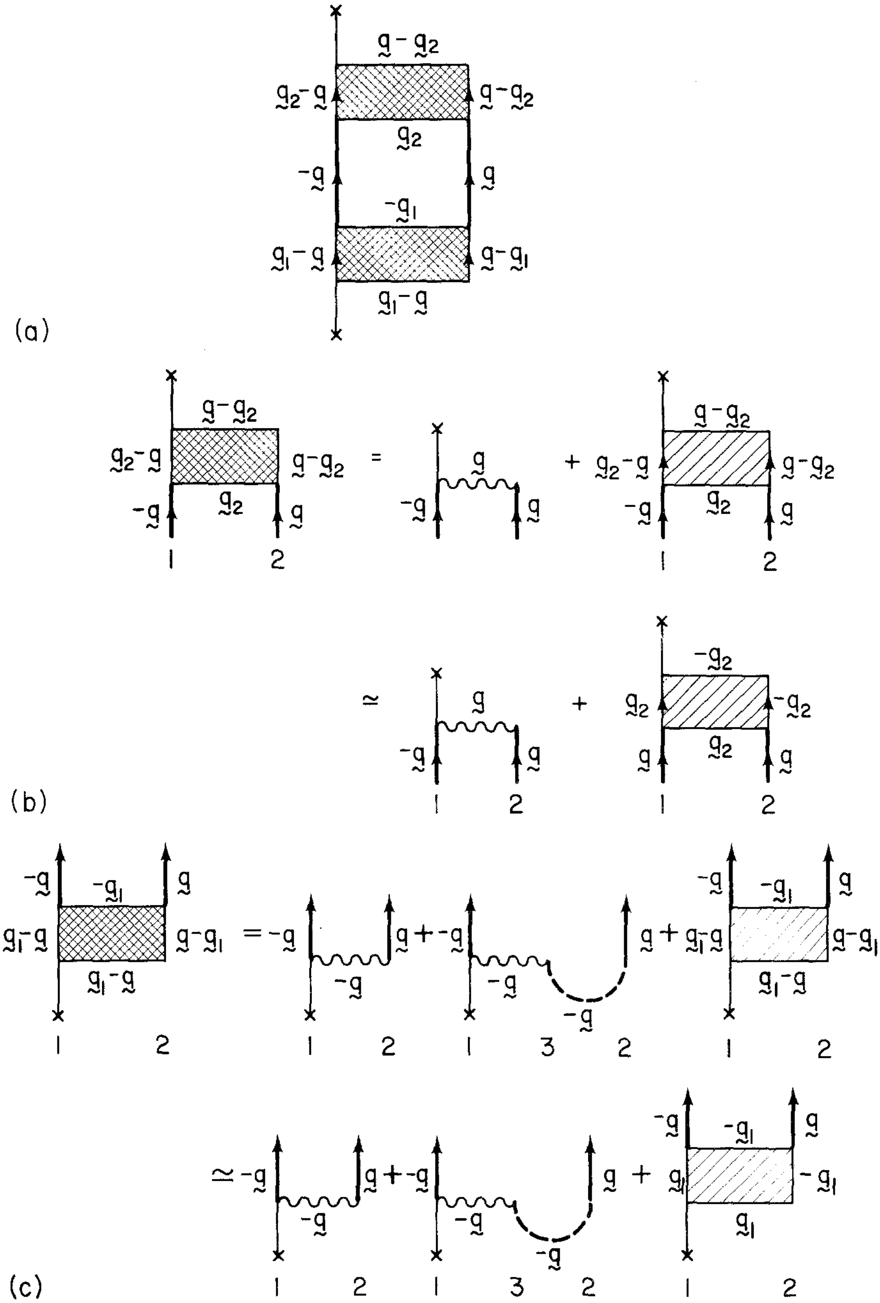

Fig. 3. (a) General two-mode diagram; (b) left (top) vertex function; (c) right (bottom) vertex function.

equality in Figs. 3b, c we have also anticipated that since for small z the relevant contribution will come from the small-q region (here q is the variable that labels the hydrodynamic modes of the renormalized propagators), the vertex functions can be approximated with their $q \to 0$ limit as long as they are nonsingular in such limit. The operators T_s^R and T_s^L corresponding to strong collisions are indeed well behaved as $q \to 0$. This is not, however, true for $\theta_{\mathbf{q}}$ and $\hat{\theta}_{\mathbf{q}}$.

We now outline the procedure used to systematically identify all the collision sequences contributing to the right and left vertices of the two-mode diagram of Fig. 3. Since we are assuming the interaction potential to be weak, we expand both the dynamical operators and the static correlation functions in Eqs. (2.8) in powers of the potential. We then analyze in detail the collision sequences contained in each operator B_2, B_3, etc., and develop an ordering scheme that enables us to determine T_s^R and T_s^L.

In order to exhibit the general structure of the operators B_l, it is often convenient to shift the location of the static correlation functions with respect to the propagators. The shifting rules are obtained by inserting the density expansion of the equilibrium distribution functions, given in Eq. (2.5c), in the equilibrium hierarchy and equating terms of the same order in the density. For instance, to lowest order in the density one obtains,

$$L_s(1,..., s)\, f_s^{(0)}(1,..., s) = 0 \tag{3.1}$$

with $f_s^{(0)}$ given by Eq. (2.5d) for $l = 0$.

The analysis of B_2 is straightforward. Using Eq. (3.1) for $(s = 2)$ with

$$f_2^{(0)}(1, 2) = \phi(v_1)\, \phi(v_2)(1 + f_{12}) \tag{3.2a}$$

where f_{ij} is the Mayer f functions for particles i and j, B_2 can immediately be rewritten in the form

$$nB_2(1, z) = n\phi^{-1}(v_1) \int d2\theta_{12} G(12)\, \phi(v_1)\, \phi(v_2)(1 + f_{12})\theta_{12} \tag{3.2a}$$

Both the propagators $G(12)$ and the Mayer function f_{12} are then expanded in powers of the potential, as discussed in Ref. 7. In Figs. 4a, b, c we show the diagrams corresponding to the terms of such an expansion of B_2 that contribute to $\theta G_s \theta$, $T_s^L G_s \theta$ and $\theta G_s T_s^R$, and $T_s^L G_s T_s^R$, respectively. We note that Fig. 4a actually contributes to Λ_s^0 so that it is subtracted in Eq. (2.13). Further, we shall see below that to construct both $T_s^L G_s \theta$ and $\theta G_s T_s^R$ we need two of the diagrams given by Fig. 4b. The addition of diagrams

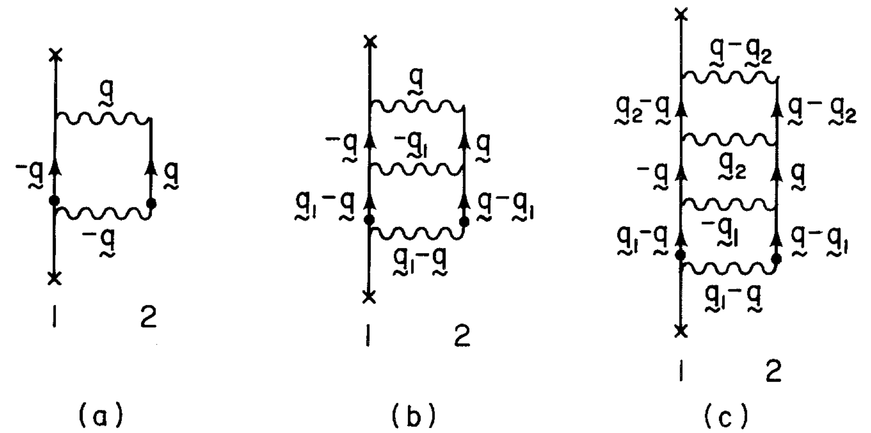

Fig. 4. (a) Contribution to $\theta G_s \theta$ from B_2; (b) contribution to $\theta G_s T_s^R$ and $T_s^L G_s \theta$ from B_2; (c) contribution to $T_s^L G_s T_s^R$ from B_2.

without renormalized intermediate propagators is of no consequence for the LTT. It follows from the ordering scheme given below that the terms in B_2 containing static or equilibrium correlations can be neglected in constructing the LTT ring operator for moderately dense OCP. Figure 4c defines the contribution of $O(n^0)$ to T_s^L and T_s^R, denoted with $T_s^{R,L(0)}$ and given by

$$T_s^{R(0)}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) = \int \frac{d\mathbf{q}_1}{(2\pi)^3} \theta_{-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)[z + i(\mathbf{q}_1 - \mathbf{q}) \cdot \mathbf{v}_{12}]^{-1} \times [\theta_{\mathbf{q}_1 - \mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)] \tag{3.3}$$

The left vertex $T_s^{L(0)}$ is given by a similar expression. We note that $T_s^{R(0)}$ and $T_s^{L(0)}$ diverge logarithmically as $q, z \to 0$. Correspondingly, we classify $T_s^{R(0)}$ and $T_s^{L(0)}$ as contributions of $O(n^0 q^0)$. In analogy with I we expect T_s^R and T_s^L to be given by an infinite series where the lth term, arising from B_l, diverges as $[n/q^2]^{l-2}$ ($l = 2, 3,...$). The terms of $O(n/q^2)$ can then be identified by analyzing B_3.

The analysis of B_3 is considerably more complicated than that of B_2. There are three distinct classes of terms: (1) diagrams that begin to renormalize the free propagators in the diagrams of Fig. 4 to the renormalized propagators given in Fig. 2b[5]; (2) diagrams that determine the first-order

[5] It should be remarked that the resummation to renormalized two-particle propagators given by (2.13b) is most conveniently performed in time language since only then do the propagators for tagged and fluid particle propagation commute.

corrections to the vertices T_s^L and T_s^R; (3) diagrams which do not contribute to the LTT to leading order in the plasma parameter. (This includes diagrams that represent the coupling of more than two modes.)

As done for B_2, to write B_3 in a convenient form we use Eq. (3.1) together with the following identity:

$$f_2^{(1)}(1,2)\,G_0^{-1}(12)=G_0^{-1}(12)\,f_2^{(1)}(1,2)+\int d3(\bar{\theta}_{13}+\bar{\theta}_{23})\,f_3^{(0)}(1,2,3) \qquad (3.4a)$$

where

$$G_0(12)=[z+L_0(1)+L_0(2)]^{-1} \qquad (3.4b)$$

Equation (3.4a) follows directly from the equilibrium hierarchy equations for the equilibrium distribution functions to first order in the density. The collision operator B_3 can then be written as

$$\begin{aligned} n^2B_3(1,z)=n^2\phi^{-1}(v_1)\int d2d3\theta_{12}G(12)[&f_{23}(1+f_{12})\,\phi(v_2)\,f_2^{(0)}(13)\theta_{13}\\ &+\theta_{13}f_{23}(1+f_{12})\,G(13)\,\phi(v_2)\,f_2^{(0)}(13)\theta_{13}\\ &+(\theta_{13}+\theta_{23})\,G(123)\,f_3^{(0)}(123)[(\theta_{13}+\theta_{23})\,G(12)\theta_{12}\\ &+(\theta_{12}+\theta_{23})\,G(13)\theta_{13}]\\ &+(\theta_{13}+\theta_{23})(1+f_{23})(1+f_{12})\,\phi(v_2)\,G(13)\,f_2^{(0)}(13)\theta_{13}\\ &+f_{13}f_{32}f_2^{(0)}(12)\,\phi(v_3)\theta_{12}] \end{aligned} \qquad (3.5a)$$

where

$$\begin{aligned} f_3^{(0)}(1,2,3)=\phi(v_1)\,\phi(v_2)\,\phi(v_3)[&1+f_{12}+f_{13}+f_{23}+f_{12}f_{13}\\ &+f_{12}f_{23}+f_{13}f_{23}+f_{12}f_{13}f_{23}] \end{aligned} \qquad (3.5b)$$

A detailed analysis of the expansion of Eq. (3.5a) in powers of the potential allows us to identify the contribution of $O(n/q^2)$ to T_s^R and T_s^L for $z\to 0$ and $q\to 0$. The relevant contributions are given in diagrammatic form in Figs. 5a, b, respectively. In B_3 one also finds the first diagrams that are needed to construct the renormalized tagged particle and fluid particle propagators of Figs. 2a and 2b, respectively. Finally, in B_3 we neglect diagrams that involve the coupling of more than two hydrodynamic modes or that represent corrections to $T_s^{R,L}$ that are less divergent than n/q^2.

The four-particle collision operator B_4 can be analyzed in a similar way. Because of its complexity we discuss only its structure here. First, it contains diagrams that contribute to the propagators in Fig. 2. Secondly, it

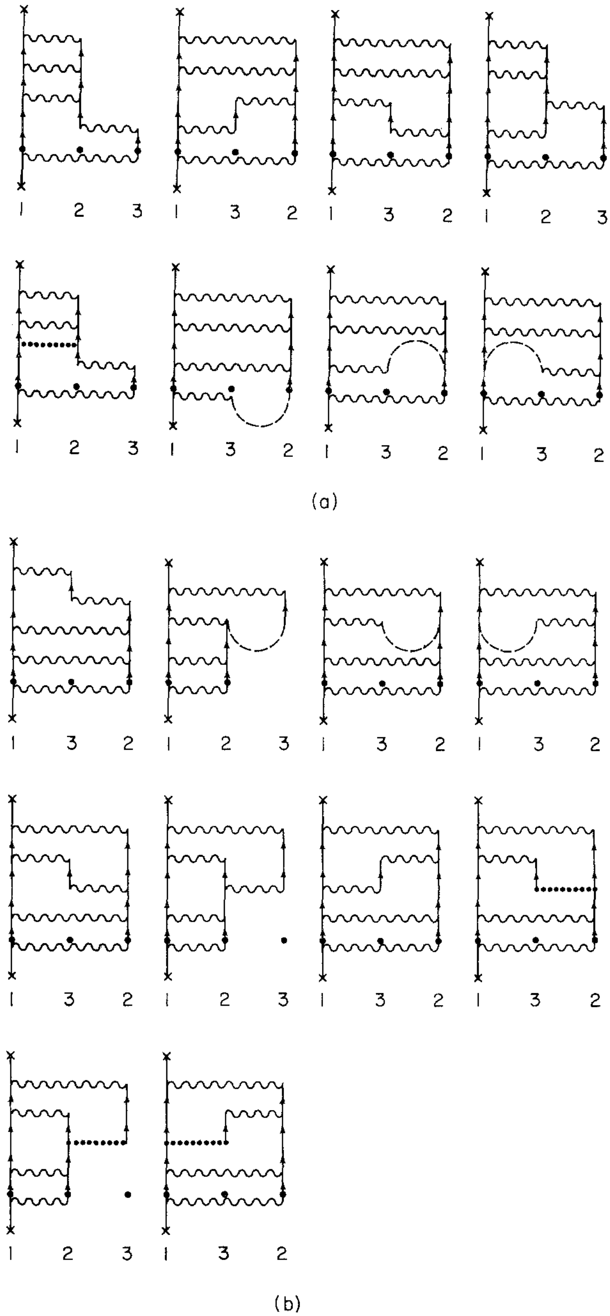

Fig. 5. Contributions to $T_s^L G_s T_s^R$ from B_3.

contains diagrams that represent correction of $O([n/q^2]^2)$ to $T_s^{R,L}$. The latter class of diagrams has, however, a particularly simple structure. Their role is simply to start the renormalization of the static bonds in Figs. 4a, b to Debye–Hückel pair correlation functions and the resummation of the fluid particle propagators to Vlasov propagators. We assume that this remains true to every order in n/q^2, i.e., that in each B_l, for $l \geqslant 5$, there are no topologically new most singular corrections to the top and bottom vertices. Each B_l, for $l \geqslant 4$, also contains terms that do not contribute to the two-mode coupling diagram or are responsible for corrections to the vertices that are less singular than those considered here.

We can now identify the top and bottom vertices T_s^R and T_s^L. We will not, however, give here their exact expression explicitly. Instead, we observe that to evaluate the contribution of the two-mode diagram given in Fig. 3 to the long time tail of the VACF, we will, in the limit $z \to 0$, use a spectral representation of the intermediate propagators on the basis of their hydrodynamic modes (cf. Section 4). As a consequence, the collision vertices T_s^R and T_s^L will always act on hydrodynamic eigenfunctions, that, to lowest order in q, are a linear combination of the five conserved densities. Specifically, when evaluating the long time tails we will only need the limiting form of $T_s^{R,L}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$ for $\mathrm{Re} z \to 0+$ when acting on the unit function for label 1 and on the five conserved densities $\psi_\alpha(\mathbf{v}_2) = \{1, \mathbf{v}_2, v_2^2\}$ for label 2. Such limiting forms are defined by

$$\lim_{\mathrm{Re} z \to 0^+} \int d\mathbf{v}_1 \mathbf{v}_2 A(\mathbf{v}_1)\, T_{s,H}^L(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, \phi(v_1)\, \phi(v_2)\, \psi_\alpha(\mathbf{v}_2)$$

$$= \lim_{\mathrm{Re} e \to 0} + \int d\mathbf{v}_1\, d\mathbf{v}_2 A(\mathbf{v}_1)\, T_s^L(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, \phi(v_1)\, \phi(v_2)\, \psi_\alpha(\mathbf{v}_2) \quad (3.6a)$$

and

$$\lim_{\mathrm{Re} z \to 0^+} \int d\mathbf{v}_1\, d\mathbf{v}_2\, \psi_\alpha(\mathbf{v}_2)\, T_{s,H}^R(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, \phi(v_1)\, \phi(v_2)\, A(\mathbf{v}_1)$$

$$= \lim_{\mathrm{Re} z \to 0^+} \int d\mathbf{v}_1\, d\mathbf{v}_2\, \psi_\alpha(\mathbf{v}_2)\, T_s^R(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, \phi(v_1)\, \phi(v_2)\, A(\mathbf{v}_1), \quad (3.6b)$$

where $A(\mathbf{v})$ is an arbitrary function. The explicit form of $T_{s,H}^R$ and $T_{s,H}^L$ is given in diagrammatic notation in Figs. 6a and 6b (the diagrams containing $\bar{\theta}$ bonds are not shown).

Their analytic expression is

$$T_{s,H}^R(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$$

$$= \int \frac{d\mathbf{q}_1}{(2\pi)^3}\, \theta_{\mathbf{q}_1 - \mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\, G_s^V(\mathbf{v}_1, \mathbf{v}_2)[\hat{\theta}_{-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) + \hat{\bar{\theta}}_{-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)] \quad (3.7a)$$

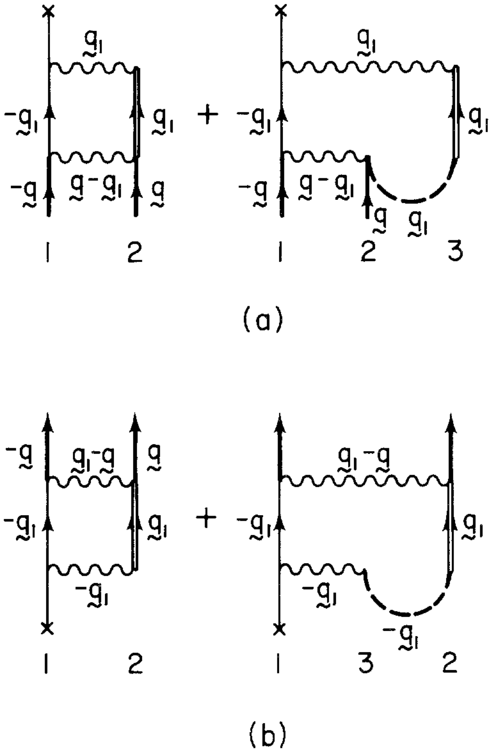

Fig. 6. (a) Diagrammatic representation of $T_{s,H}^{L}$, given by Eq. (3.7b); (b) diagrammatic representation of $T_{s,H}^{R}$, given by Eq. (3.7a). In both (a) and (b) the diagrams containing $\bar{\theta}$ bonds are now shown.

and

$$T_{s,H}^{L}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) = [T_{s,H}^{R}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)]^{\dagger}$$
$$= \int \frac{d\mathbf{q}_1}{(2\pi)^3} \Bigg[\theta_{\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)\, G_s^V(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}_1, z)$$
$$+ n \int d\mathbf{v}_3 \theta_{\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)\, G_s^V(\mathbf{v}_1, \mathbf{v}_3, \mathbf{q}_1, z)\, h_{\mathrm{DH}}(q_1)\, \phi(v_3) \Bigg]$$
$$\times\ [\theta_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) + \bar{\theta}_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)] \tag{3.7b}$$

where

$$G_s^V(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) = [z - i\mathbf{q}\cdot\mathbf{v}_{12} - \mathscr{V}_{\mathbf{q}}(\mathbf{v}_2)]^{-1} \tag{3.7c}$$

and $\hat{\bar{\theta}}_{\mathbf{q}}$ is defined, in analogy with $\bar{\theta}_{\mathbf{q}}$, as

$$\hat{\bar{\theta}}_{\mathbf{q}}(\mathbf{v}_i, \mathbf{v}_j) = -\phi^{-1}(v_i)\, \phi^{-1}(v_j) [\hat{\theta}_{\mathbf{q}}(\mathbf{v}_i, \mathbf{v}_j)\, \phi(v_i)\, \phi(v_j)] \tag{3.7d}$$

The identity $T^L_{s,H} = [T^R_{s,H}]^\dagger$ is proved in Appendix B. We have shown that $T^R_{s,H}$ and $T^L_{s,H}$ are indeed given by Eqs. (3.7a, b) by evaluating explicitly the right-hand side of Eqs. (3.6a, b) using techniques analogous to those employed in Appendix B to prove the properties of $T^{R,L}_{s,H}$. An alternative proof is also provided by evaluating term by term the density expansion of the right-hand side of Eqs. (3.6a, b) and verifying that the result is to each order identical to the density expansion of $T^R_{s,H}$ and $T^L_{s,H}$.

We can now write the explicit form of the four ring-like collision operators discussed earlier. Denoting these operators by $\Delta B_{\theta\theta}$, already given by Eq. (2.13), $\Delta B_{T\theta}$, $\Delta B_{\theta T}$ and ΔB_{TT}, we have

$$\Delta B_{T\theta}(\mathbf{v}_1, z) = n\phi^{-1}(v_1) \int \frac{d\mathbf{q}}{(2\pi)^3} T^L_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) \times \phi(v_1)\, \phi(v_2)\, \hat{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \tag{3.8a}$$

and

$$\Delta B_{\theta T}(\mathbf{v}_1, z) = n\phi^{-1}(v_1) \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\, G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) \times T^R_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, \phi(v_1)\, \phi(v_2) \tag{3.8b}$$

and

$$\Delta B_{TT}(\mathbf{v}_1, z) = n\phi^{-1}(v_1) \int \frac{d\mathbf{q}}{(2\pi)^3} T^L_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) \times T^R_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, \phi(v_1)\, \phi(v_2) \tag{3.8c}$$

Combining Eqs. (3.8), (2.12), and (2.1) we also define

$$C^{(1)}_{D_{\alpha\beta}}(z) = \int dv_1\, \phi(v_1)\, v_{1x} [z - \Lambda^0_s(\mathbf{v}_1, z)]^{-1} \Delta B_{\alpha\beta}(\mathbf{v}_1) [z - \Lambda^0_s(\mathbf{v}_1, z)]^{-1} v_{1x} \tag{3.9}$$

with α, $\beta = \theta$ or T.

4. EVALUATION OF THE LONG TIME TAILS

In the first part of this section we list the hydrodynamic modes of the BGL propagators defined in Eq. (2.13b). The hydrodynamic eigenvalues and eigenfunctions have been derived elsewhere[11,16] and are needed here to determine the long time behavior of the VACF.

Next, we evaluate explicitly the long time, or small-z, limit of Eq. (3.9) for $\alpha = \beta = T$. We obtain a $t^{-3/2}$ long time tail for the VACF, $C_D(t)$, in agreement with the results of the phenomenological mode coupling theory.

The analogous LTT results for the kinetic parts of the correlation functions that determine the shear viscosity, η, and the heat conductivity, λ, are quoted.

Finally, it can be shown that the contributions to the VACF given by Eq. (3.9) for $(\alpha, \beta) = ((\theta, \theta), (\theta, T), (T, \theta))$ (i.e., corresponding to the case when the first and/or the last collision in the ring operator are weak collisions) only contain oscillating long time tails of the type given in Eq. (1.1). The same holds true for the other transport coefficients. Below we argue that these oscillating LTT should be neglected in a consistent theory.

4.1. The Hydrodynamic Modes

To derive an explicit representation for Eq. (2.13b) we make a spectral decomposition of the two propagators in $G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$. Since we are interested only in the long time behavior of $C_D(t)$ we can restrict ourselves to the low-lying or hydrodynamic modes of these operators. These modes are distinguished by the vanishing of their damping as q^2 for $q \to 0$. The eigenvalue problems to be solved for tagged particle motion and fluid particle motion are, respectively,

$$[i\mathbf{q} \cdot \mathbf{v}_1 - \tilde{\Lambda}_s^0(\mathbf{v}_1, -\omega_s)] |\Theta_s^R(\mathbf{v}_1, \mathbf{q})\rangle_\phi = \omega_s |\Theta_s^R(\mathbf{v}_1, \mathbf{q})\rangle_\phi \tag{4.1a}$$

and

$$[i\mathbf{q} \cdot \mathbf{v}_2 - \mathbf{v}_\mathbf{q}(\mathbf{v}_2) - \tilde{\Lambda}(\mathbf{v}_2, \mathbf{q}, -\omega_\alpha)] |\Theta_\alpha^R(\mathbf{v}_2, \mathbf{q})\rangle_\phi = \omega_\alpha |\Theta_\alpha^R(\mathbf{v}_2, \mathbf{q})\rangle_\phi \tag{4.1b}$$

Here we have used that for small values of the plasma parameter we can neglect the q dependence of the collision operators. In Eqs. (4.1a, b) we have introduced a bra and ket notation and defined an inner product in velocity space as

$$\langle f \mid g \rangle_\phi = \int d\mathbf{v} f(\mathbf{v})\, g(\mathbf{v})\, \phi(v) \tag{4.1c}$$

for any f and g. In this notation the weight function is in the ket vector, as indicated. In Eqs. (4.1a, b) ω_s is the hydrodynamic eigenvalue for tagged particle motion (self-diffusion) and ω_α denotes the five hydrodynamic eigenvalues for fluid particle motion; the Θ_α^R are the corresponding right eigenfunctions. Since these eigenvalue problems are not self-adjoint there are also analogous left eigenvalue problems with eigenfunctions Θ_α^L.

For small q the above eigenvalue problems can be easily solved by a standard technique.[6] For the self-eigenvalue problem one obtains

$$\omega_s(q) = Dq^2 + O(q^4) \tag{4.2a}$$

[6] The solution of the eigenvalue problem is based on the introduction of an operator that projects on the space spanned by the quantities that are conserved in a binary collision. A description of the method, used there in a different contest, can be found for example in Ref. 17.

and

$$\Theta_s^R(\mathbf{q}, \mathbf{v}_1) = [\Theta_s^L(\mathbf{q}, \mathbf{v}_1)]^* = 1 + \frac{1}{\Lambda_s^0(\mathbf{v}_1, 0)} i\mathbf{q}\cdot\mathbf{v}_1 + O(q^2) \tag{4.2b}$$

where the asterisk denotes the complex conjugate. In Eq. (4.2a), D is the self-diffusion coefficient to lowest order in ε_p,

$$D = -\langle \hat{\mathbf{q}}\cdot\mathbf{v}_1 | \frac{1}{\tilde{\Lambda}_s^0(\mathbf{v}_1, 0)} | \hat{\mathbf{q}}\cdot\mathbf{v}_1 \rangle_\phi \tag{4.2c}$$

where $\hat{\mathbf{q}} = \hat{\mathbf{q}}/|\mathbf{q}|$.

The fluid hydrodynamic eigenvalue problem is also easily solved. There are five long wavelength modes in the fluid, corresponding to the five conserved densities: two shear modes, one heat mode and two plasma modes. The eigenvalue and eigenfunctions for the shear modes are

$$\omega_{\nu_i}(q) = \omega_\nu(q) = \nu q^2 + O(q^4) \tag{4.3a}$$

and

$$\begin{aligned} \Theta_{\nu_i}^R(\mathbf{q}, \mathbf{v}_2) &= [\Theta_{\nu_i}^L(\mathbf{q}, \mathbf{v}_2)]^* \\ &= (m\beta)^{1/2} \left[\hat{\mathbf{q}}^i\cdot\mathbf{v}_2 + iq_\alpha \hat{q}^i_\beta \frac{1}{\Lambda(\mathbf{v}_2, 0, 0)} v_{2\alpha} v_{2\beta} \right] + O(q^2) \end{aligned} \tag{4.3b}$$

for $i = 1, 2$. Here $(\hat{\mathbf{q}}, \hat{\mathbf{q}}^1, \hat{\mathbf{q}}^2)$ form a set of orthonormal unit vectors, and the summation convection has been used. In Eq. (4.3a), ν is the zero-frequency kinematic viscosity, given by

$$\nu = -m\beta \langle \hat{\mathbf{q}}\cdot\mathbf{v}_2 \hat{\mathbf{q}}^i\cdot\mathbf{v}_2 | \frac{1}{\tilde{\Lambda}(\mathbf{v}_2, 0, 0)} | \hat{\mathbf{q}}\cdot\mathbf{v}_2 \hat{\mathbf{q}}^i\cdot\mathbf{v}_2 \rangle_\phi \tag{4.3c}$$

There is a single heat or entropy mode, with eigenvalue

$$\omega_H(q) = D_T q^2 + O(q^4) \tag{4.4a}$$

and eigenfunctions

$$\Theta_H^R(\mathbf{q}, v_2) = \left(\frac{2}{3}\right)^{1/2} \left[\frac{1}{2} m\beta v_2^2 - \frac{3}{2} - \frac{1}{\Lambda(\mathbf{v}_2, 0, 0)} i\mathbf{q}\cdot\mathbf{v}_2 \left(\frac{1}{2} m\beta v_2^2 - \frac{5}{2} \right) + O(q^2) \right] \tag{4.4b}$$

and

$$\Theta_H^L(\mathbf{q}, \mathbf{v}_2) = \left(\frac{2}{3}\right)^{1/2} \left[\frac{1}{2} m\beta v_2^2 - \frac{5}{2} - \frac{1}{\Lambda(\mathbf{v}_2, 0, 0)} i\mathbf{q}\cdot\mathbf{v}_2 \left(\frac{1}{2} m\beta v_2^2 - \frac{5}{2} \right) + O(q^2) \right] \tag{4.4c}$$

Here $D_T = \lambda/\rho C_v$, with $C_v = C_v^0 = 3k_B/2m$ the specific heat per unit mass of an ideal gas, is the zero-frequency thermal diffusivity given by

$$D_T = -\frac{2}{3}\left\langle \hat{\mathbf{q}}\cdot\mathbf{v}_2\left(\frac{1}{2}m\beta v_2^2 - \frac{5}{2}\right)\right| \frac{1}{\tilde{\Lambda}(\mathbf{v}_2, 0, 0)} \left| \hat{\mathbf{q}}\cdot\mathbf{v}_2\left(\frac{1}{2}m\beta v_2^2 - \frac{5}{2}\right)\right\rangle_\phi \quad (4.4d)$$

Finally, there are two high-frequency plasma modes—the imaginary part of these eigenvalues does not vanish as $q \to 0$—that take the place of the sound modes of a neutral fluid. The eigenvalues are given by

$$\omega_\sigma(q) = i\sigma\omega_p\left[1 + \frac{q^2}{2\omega_p}\left(\frac{c^2}{\omega_p} + \gamma_p\right)\right] + \frac{1}{2}\Gamma_p q^2 + O(q^4) \quad (4.5a)$$

for $\sigma = \pm 1$, and the corresponding eigenfunctions are

$$\Theta_\sigma^R(\mathbf{q}, \mathbf{v}_2) = \left(\frac{m\beta}{2}\right)^{1/2}\left[\sigma\hat{\mathbf{q}}\cdot\mathbf{v}_2 + \frac{(m\beta)^{1/2}}{3} q\lambda_D v_2^2 + i\sigma q \frac{1}{i\sigma\omega_p + \Lambda(\mathbf{v}_2, 0, -i\sigma\omega_p)} \hat{q}_\alpha \hat{q}_\beta \left(v_{2\alpha}v_{2\beta} - \frac{1}{3}\delta_{\alpha\beta}v_2^2\right) + O(q^2)\right] \quad (4.5b)$$

and

$$\Theta_\sigma^L(\mathbf{q}, \mathbf{v}_2) = \left(\frac{m\beta}{2}\right)^{1/2}\left[\sigma\hat{\mathbf{q}}\cdot\mathbf{v}_2) + \frac{(m\beta)^{-1/2}}{q\lambda_D} + \frac{(m\beta)^{1/2}}{3} q\lambda_D v_2^2 + q\left(\frac{c^2}{2\omega_p} + \frac{1}{2}\gamma_p - \frac{i\sigma}{2}\Gamma_p\right) - i\sigma q \frac{1}{i\sigma\omega_p + \Lambda(\mathbf{v}_2, 0, -i\sigma\omega_p)} \hat{q}_\alpha \hat{q}_\beta \left(v_{2\alpha}v_{2\beta} - \frac{1}{3}\delta_{\alpha\beta}v_2^2\right) + O(q^2)\right] \quad (4.5c)$$

Here c^2 is the low-density speed of sound, $c^2 = 5k_B T/3m$. In Eq. (4.5a)

$$\gamma_p = 2\sigma \operatorname{Im} K_p, \qquad \Gamma_p = 2 \operatorname{Re} K_p \quad (4.5d)$$

with

$$K_p = -m\beta\left\langle\left[(\hat{\mathbf{q}}\cdot\mathbf{v}_2)^2 - \frac{v_2^2}{3}\right]\right| \frac{1}{i\sigma\omega_p + \tilde{\Lambda}(\mathbf{v}_2, 0, -i\sigma\omega_p)} \left|\left[(\mathbf{q}\cdot\mathbf{v}_2)^2 - \frac{v_2^2}{3}\right]\right\rangle_\phi \quad (4.5e)$$

We remark that the terms of $O(q)$ in the hydrodynamic eigenfunctions will not contribute to the LTT obtained from Eq. (3.6) for $\alpha = \beta = T$. This is also the case for neutral fluids. We give these correction terms here because they are needed to consistently establish that Eq. (3.6) for α and/or $\beta = \theta$ does not lead to an asymptotic LTT.

4.2. Evaluation of $C_D(t)$ for Long Times

An explicit representation of Eq. (2.13b) that applies for small z or long times is obtained by keeping only the hydrodynamic modes in the spectral decomposition of the two propagators in $G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$. The contribution from nonhydrodynamic modes can easily be shown to lead to an exponential decay of the time correlation functions. Similarly, branch point singularities do not contribute to the leading asymptotic long time tails.[16] For small z, the spectral representation of the two-particle propagator $G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$ is approximated by

$$G_s(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) \approx \sum_{\alpha = \nu,\sigma,H} |\Theta_s^R(\mathbf{v}_1, -\mathbf{q})\, \Theta_\alpha^R(\mathbf{v}_2, \mathbf{q})\rangle_\phi \times [z + \omega_s(q) + \omega_\alpha(q)]^{-1} \langle \Theta_s^L(\mathbf{v}_1, -\mathbf{q})\, \Theta_\alpha^L(\mathbf{v}_2, \mathbf{q})| \tag{4.6}$$

where the sum is restricted to the five hydrodynamic modes. For small z, the contribution to the VACF arising from the class of collision sequences contained in the ring operator ΔB_{TT} is then obtained by combining Eqs. (3.9), (3.5c), and (4.6), with the result

$$\begin{aligned} C_{DTT}^{(1)}(e) = {} & \sum_{\alpha = \nu,\sigma,H} n \int \frac{d\mathbf{q}}{(2\pi)^3} \langle v_{1x}| [z - \tilde{\Lambda}_s^0(\mathbf{v}_1, z)]^{-1} \\ & \times T_{s,H}^L(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) |\Theta_s^R(\mathbf{v}_1, -\mathbf{q})\, \Theta_\alpha^R(\mathbf{v}_2, \mathbf{q})\rangle_\phi [z + \omega_s(q) + \omega_\alpha(q)]^{-1} \\ & \times \langle \Theta_s^L(\mathbf{v}_1, -\mathbf{q})\, \Theta_\alpha^L(\mathbf{v}_2, \mathbf{q})| T_{s,H}^R(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) [z - \tilde{\Lambda}_s^0(\mathbf{v}_1, z)]^{-1} |v_{1x}\rangle_\phi \\ = {} & \sum_{\alpha = \nu,\sigma,H} n \int \frac{d\mathbf{q}}{(2\pi)^3} \langle v_{1x}| [\tilde{\Lambda}_s^0(\mathbf{v}_1, 0)]^{-1} \\ & \times T_{s,0}^L(\mathbf{v}_1, \mathbf{v}_2) |\Theta_s^R(\mathbf{v}_1, \mathbf{q} = 0)\, \Theta_{\alpha,0}^R(\mathbf{v}_2, \mathbf{q})\rangle_\phi [z + \omega_s(q) + \omega_\alpha(q)]^{-1} \\ & \times \langle \Theta_s^L(\mathbf{v}_1, \mathbf{q} = 0)\, \Theta_{\alpha,0}^L(\mathbf{v}_2, \mathbf{q})| T_{s,0}^R(\mathbf{v}_1, \mathbf{v}_2) [\tilde{\Lambda}_s^0(\mathbf{v}_1, 0)]^{-1} |v_{1x}\rangle_\phi \end{aligned} \tag{4.7a}$$

where

$$T_{s,0}^{R,L}(\mathbf{v}_1, \mathbf{v}_2) = \lim_{\mathrm{Re}\, z \to 0^+} \lim_{q \to 0} T_{s,H}^{R,L}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) \tag{4.7b}$$

and $\Theta^R_{\alpha,0}(\mathbf{v}, \mathbf{q})$ and $\Theta^L_{\alpha,0}(\mathbf{v}, \mathbf{q})$ denote the eigenfunctions up to order q^0 [these do not necessarily coincide with $\Theta^{R,L}_\alpha(\mathbf{v}, \mathbf{q}=0)$, since Θ^L_α contains terms of $O(1/q)$]. In writing the last equality in Eq. (4.7a) we have used that in the $z \to 0$ limit the dominant contribution to $C^{(1)}_{D_{TT}}(z)$ comes from the $q \to 0$ region of the $\mathbf{q}$ integration. The T operators are nonsingular functions in the limit q, $z \to 0$.

In Appendix A we show that

$$\begin{aligned}\langle v_{1x} | [\tilde{\Lambda}^0_s(\mathbf{v}_1, 0)]^{-1} T^L_{s,0}(\mathbf{v}_1, \mathbf{v}_2) | \Theta^R_s(\mathbf{v}_1, \mathbf{q}=0)\, \Theta^R_{\alpha,0}(\mathbf{v}_2, \mathbf{q}) \rangle_\phi \\ = -n^{-1} \langle v_{1x} | \Theta^R_{\alpha,0}(\mathbf{v}_1, \mathbf{q}) \rangle_\phi \end{aligned} \tag{4.8a}$$

and

$$\begin{aligned}\langle \Theta^L_s(\mathbf{v}_1, \mathbf{q}=0)\, \Theta^L_{\alpha,0}(\mathbf{v}_2, \mathbf{q}) | T^R_{s,0}(\mathbf{v}_1, \mathbf{v}_2) [\tilde{\Lambda}^0_s(\mathbf{v}_1, 0)]^{-1} | v_{1x} \rangle_\phi \\ = -n^{-1} \langle \Theta^L_{\alpha,0}(\mathbf{v}_1, \mathbf{q}) | v_{1x} \rangle_\phi \end{aligned} \tag{4.8b}$$

Similar identities are also valid for neutral fluids.[18] Finally, making use of Eqs. (4.8) as well as of the explicit form of the hydrodynamic eigenfunctions and eigenvalues, given in Eqs. (4.2)–(4.5), and transforming to a time representation, we obtain

$$\begin{aligned} C^{(1)}_{D_{TT}}(t) &\approx \sum_{\alpha = \nu,\sigma,H} \int \frac{d\mathbf{q}}{(2\pi)^3} \langle v_{1x} | \Theta^R_{\alpha,0}(\mathbf{v}_1, \mathbf{q}) \rangle_\phi \langle \Theta^L_{\alpha,0}(\mathbf{v}_1, \mathbf{q}) | v_{1x} \rangle_\phi \\ &\quad \times \exp\{-t[\omega_s(q) + \omega_\alpha(q)]\} \\ &= \frac{2}{3\beta n m} \left\{ \frac{1}{[4\pi t(D+\nu)]^{3/2}} + \frac{1}{(4\pi t)^{3/2}} \right. \\ &\quad \left. \times \frac{\cos(\omega_p t + \delta)}{[(D + \frac{1}{2}\Gamma_p)^2 + (c^2/2\omega_p + \frac{1}{2}\gamma_p)^2]^{3/4}} + O(1/t^{5/2}) \right\} \end{aligned} \tag{4.9a}$$

with

$$\delta = \frac{3}{2} \tan^{-1} \left(\frac{c^2/2\omega_p + \gamma_p/2}{D + \Gamma_p/2} \right) \tag{4.9b}$$

Equation (4.9) is identical to the result obtained for the LTT of $C_D(t)$ using phenomenological mode coupling theory.[9,10]

The oscillatory LTT contribution in Eq. (4.9a) approximately averages to zero for long times and can be neglected compared to the first term in Eq. (4.9a). This first term is identical to the result obtained for neutral fluids.[11]

4.3. Long Time Behavior of $C_\eta(t)$ and $C_\lambda(t)$

The time correlation functions that determine the kinetic parts of the shear viscosity and heat conductivity are, respectively,

$$C_\eta(t) = \lim_{\substack{N,\Omega\to\infty\\ N/\Omega=n}} \frac{1}{\Omega}\left\langle \sum_{i=1}^{N} j_\eta(\mathbf{v}_i(t)) \sum_{j=1}^{N} j_\eta(\mathbf{v}_j)\right\rangle$$

$$= n\int d\mathbf{v}_1\, \phi(v_1)\, j_\eta(\mathbf{v}_1)\, \Phi_\eta(\mathbf{v}_1, t) \tag{4.10a}$$

and

$$C_\lambda(t) = \lim_{\substack{N,\Omega\to\infty\\ N/\Omega=n}} \frac{1}{\Omega}\left\langle \sum_{i=1}^{N} j_\lambda(\mathbf{v}_i(t)) \sum_{j=1}^{N} j_\lambda(\mathbf{v}_j)\right\rangle$$

$$= n\int d\mathbf{v}_1\, \phi(v_1)\, j_\lambda(\mathbf{v}_1)\, \Phi_\lambda(\mathbf{v}_1, t) \tag{4.10b}$$

where the one-particle currents j_η and j_λ are given by

$$j_\eta(\mathbf{v}_1) = v_{1x} v_{1y} \tag{4.10c}$$

and

$$j_\lambda(\mathbf{v}_1) = v_{1x}\left(\frac{1}{2}m\beta v_1^2 - \frac{5}{2}\right) \tag{4.10d}$$

Kinetic equations for Φ_η and Φ_λ can be derived by following the same procedure as outlined in Section 3 for Φ_D. The details are not given here. As for C_D, the dominant LTT arise from a ring collision operator where both the first and last collisions are strong collisions. Adopting the same notation used for the VACF, $C^{(1)}_{\mu TT}(z)$ $(\mu = \eta, \lambda)$ is given by

$$C^{(1)}_{\mu TT} = n^2 \int \frac{d\mathbf{q}}{(2\pi)^3} \langle j_\mu(\mathbf{v}_1)|\,[z - \tilde{\Lambda}(\mathbf{v}_1, \mathbf{q}=0, z)]^{-1}\, T^L_H(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$$
$$\times\; G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, T^T_H(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}=0, z)[z - \tilde{\Lambda}(\mathbf{v}_1, q=0, z)]^{-1}\, |j_\mu(\mathbf{v}_1)\rangle_\phi \tag{4.11a}$$

where $G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$ is the renormalized propagator for two fluid particles [cf. Eq. (2.13b)],

$$G(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) = \int_0^\infty dt\, e^{-zt}\, \Gamma_{-\mathbf{q}}(\mathbf{v}_1, t)\, \Gamma_{\mathbf{q}}(\mathbf{v}_2, t) \tag{4.11b}$$

and $T_H^{R,L}$ represent the "hydrodynamic part" of the strong binary collision operators, given by

$$T_H^R(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) = \int \frac{d\mathbf{q}_1}{(2\pi)^3} \theta_{\mathbf{q}_1-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\, G^V(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}_1, z)(1 + P_{12} P_{\mathbf{q}_1, -\mathbf{q}_1}) \times [\hat{\theta}_{-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) + \hat{\bar{\theta}}_{-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)] \quad (4.11c)$$

where $P_{\mathbf{q},-\mathbf{q}}$ changes the sign of the wave vector $\mathbf{q}$, and

$$T_H^L(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) = \int \frac{d\mathbf{q}_1}{(2\pi)^3} \theta_{\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)\, G^V(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}_1, z) \{\theta_{\mathbf{q}-\mathbf{q}_1} + \theta_{\mathbf{q}_1-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) + n \int d\mathbf{v}_3 h_{\mathrm{DH}}(q_1)(P_{13} + P_{23})[\theta_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) + \bar{\theta}_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)]\, \phi(v_3)\} \quad (4.11d)$$

Here $G^V(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$ is a two-particle Vlasov propagator, given by

$$G^V(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z) = [z - i\mathbf{q} \cdot \mathbf{v}_{12} - \mathscr{V}_{-\mathbf{q}}(\mathbf{v}_1) - \mathscr{V}_{\mathbf{q}}(\mathbf{v}_2)]^{-1} \quad (4.11e)$$

The long time limit of Eq. (4.11a) can be evaluated as done for the VACF. Inserting two sets of fluid particle hydrodynamic modes and using identities similar to Eqs. (4.8) one obtains

$$C_{\mu TT}^{(1)}(t) \approx \frac{1}{2} \sum_{\alpha,\beta = \nu,\sigma,H} \int \frac{d\mathbf{q}}{(2\pi)^3} \langle j_\mu(\mathbf{v}_1)|\, \Theta_{\alpha,0}^R(\mathbf{v}_1, -\mathbf{q})\, \Theta_{\beta,0}^R(\mathbf{v}_1, \mathbf{q})\rangle_\phi \times \langle \Theta_{\alpha,0}^L(\mathbf{v}_1, -\mathbf{q})\, \Theta_{\beta,0}^L(\mathbf{v}_1, \mathbf{q})\, |\, j_\mu(\mathbf{v}_1)\rangle_\phi \exp\{-t[\omega_\alpha(q) + \omega_\beta(q)]\} \quad (4.12a)$$

Making use of the explicit form of the hydrodynamic eigenvalues and eigenfunctions, the long time behavior of $C_{\mu TT}^{(1)}$ is easily evaluated, with the result

$$C_{\eta TT}^{(1)} \approx \frac{7}{15\beta^2} \frac{1}{(8\pi\nu t)^{3/2}} + \frac{1}{15\beta^2} \left\{ \frac{1}{(4\pi\Gamma_p t)^{3/2}} + \frac{1}{(4\pi t)^{3/2}} \frac{\cos(2\omega_p t + \theta_p)}{[(c^2/\omega_p + \gamma_p)^2 + \Gamma_p^2]^{3/4}} \right\} + \frac{1}{5\beta^2} \frac{1}{(4\pi t)^{3/2}} \frac{\cos(\omega_p t + \theta_{\nu p})}{[(c^2/2\omega_p + \frac{1}{2}\gamma_p)^2 + (\nu + \frac{1}{2}\Gamma_p)^2]^{3/4}} + O\left(\frac{1}{t^{5/2}}\right) \quad (4.12b)$$

and

$$C_{\lambda TT}^{(1)} \approx \frac{25k_B}{6m^2C_v^0\beta^3}\left\{\frac{1}{[4\pi t(\nu+D_T)]^{3/2}}\right.$$
$$\left. + \frac{1}{2(4\pi t)^{3/2}}\frac{\cos(\omega_p t+\theta_\lambda)}{[(D_T+\frac{1}{2}\Gamma_p)^2+(c^2/2\omega_p+\frac{1}{2}\gamma_p)^2]^{3/4}} + O\left(\frac{1}{t^{5/2}}\right)\right\} \quad (4.12c)$$

where

$$\theta_p = \frac{3}{2}\tanh^{-1}\left\{\left(\frac{c^2}{\omega_p}+\gamma_p\right)\Big/\Gamma_p\right\} \quad (4.12d)$$

and

$$\theta_{\nu p} = \frac{3}{2}\tan^{-1}\left\{\frac{c^2/2\omega_p+\gamma_p/2}{\nu+\Gamma_p/2}\right\} \quad (4.12e)$$

and

$$\theta_\lambda = \frac{3}{2}\tan^{-1}\left\{\frac{c^2/2\omega_p+\gamma_p/2}{D_T+\Gamma_p/2}\right\} \quad (4.12f)$$

The contributions to $C_\mu(t)$ from ring operators where the first and/or last collisions are weak collisions can again be shown to lead only to oscillating LTT. Finally, we remark that, as discussed for $C_D(t)$, one can consistently neglect the oscillating LTT in Eqs. (4.12). We will come back to this point in the following section.

5. DISCUSSION

We conclude this paper with a few remarks.

(1) We have concentrated here on the slow long time decay of $\Phi_D^{(1)}$, given by Eq. (2.12c) and have not yet discussed the long time behavior of the solution of the BGL equations, $\Phi_D^{(0)}$, given by Eq. (2.12b). Expanding Eq. (2.12b) around the Markovian BGL operator, $\Lambda_s^0(\mathbf{v}_1, 0) = \Lambda_s^0(\mathbf{v}_1, z=0)$, we obtain

$$\Phi_D^{(0)}(\mathbf{v}_1, z) = [z-\Lambda_s^0(\mathbf{v}_1, 0)]^{-1}v_{1x} + [z-\Lambda_s^0(\mathbf{v}_1, 0)]^{-1}[\Lambda_s^0(\mathbf{v}_1, z)-\Lambda_s^0(\mathbf{v}_1, 0)]$$
$$\times [z-\Lambda_s^0(\mathbf{v}_1, 0)]^{-1}v_{1x} + \cdots \quad (5.1a)$$

The first term in Eq. (5.1) can be easily evaluated and shown to lead to an exponentially decaying VACF,

$$C_D^{(0)}(t) = (m\beta)^{-1}\exp[-t/3\sqrt{\pi}\,\tau] \quad (5.1b)$$

with

$$\tau = [\omega_p \varepsilon_p \log(\varepsilon_p^{-1})]^{-1} \tag{5.1c}$$

Here τ is the mean free time between collisions associated with the Markovian limit of the BGL operator, Eq. (2.9b).

The second term in Eq. (5.1a) was analyzed some ago by Guernsey.[19] He showed that it leads to an oscillating long time tail very similar to that given in Eq. (1.1). This behavior is very different from that of neutral fluids. The presence of a slowly decaying part of the VACF within the BGL approximation is a consequence of the long range of the Coulomb potential and is unique to the system considered here. Physically, this oscillating LTT arises in Eq. (5.1a) due to the collective nature of the Vlasov operator contained in $\Lambda_s^0(\mathbf{v}_1, z)$. In fact we showed in I that the BGL operator $\Lambda_s^0(\mathbf{v}_1, z)$ is obtained by resumming all ring-like diagrams where all the collisions are weak collisions, governed by the interaction operator θ_{ij}. Collective effects involving weak collisions are responsible for oscillating LTT in the velocity correlation functions. We argue, however, that the "true" asymptotic behavior of such velocity correlations should be identified with the purely decaying $t^{-3/2}$ tail and is therefore entirely analogous to that of neutral fluids. The oscillating LTT should be considered as decaying faster than $t^{-3/2}$ in the following sense. In an actual "experiment" one analyzes the behavior of the VACF by looking at its time average or its integral over an observation time T, with $T \gg \omega_p^{-1}$. The time average of the oscillating tail, Eq. (1.1), is then found to decay one power faster than the average of the $t^{-3/2}$ tail.

(2) Once more we want to emphasize that the only ring operator that leads to a purely decaying LTT is ΔB_{TT}, i.e., one where both the first and last collisions are strong collisions, that can correlate the velocities of the particles. Furthermore, the intermediate hydrodynamic propagators in ΔB_{TT} are obtained by resumming all the ring collisions where all the collisions of the sequence are strong. The ring operator ΔB_{TT} contains then precisely the same physical effects as the ring operator for hard sphere fluids.

In Refs. 1 and 2 a formal kinetic theory was used to obtain the ring operator responsible for the long time tails. In both cases the intermediate propagator was correctly identified as hydrodynamic propagators, built of sequences of strong collisions among the particles, but the initial and final collisions of each ring were assumed to be weak. The corresponding ring operator contains the same physical information as our $\Delta B_{\theta\theta}$, even though it differs from it in the detailed form of the vertices. Gould and Mazenko[1] only obtained an oscillating long time tail for the VACF, as given by Eq. (1.1).

From a weak-collision ring operator Baus and Wallenborn[2] have obtained at $t^{-3/2}$ contribution to the shear current autocorrelation function arising from the coupling of two plasma modes. This seems in contradiction with the conclusion we reached above that this type of ring operator does not lead to a purely algebraic long time tail of $O(t^{-3/2})$. We believe that they have not properly taken into account all the static correlations which are needed to consistently evaluate the weak-collision ring operator. Furthermore, we show elsewhere[10] that the full shear current autocorrelation function (kinetic and potential parts) that was considered by Baus and Wallenborn does not have a $t^{-3/2}$ tail arising from the coupling of two plasma modes.

(3) We remark that among the hydrodynamic modes of the OCP there are finite frequency modes, the plasma modes, for which $\operatorname{Im} \omega(\mathbf{q})$ does not vanish as $q \to 0$. As a consequence of this the transport coefficients in Eq. (4.5a) are finite frequency transport coefficients, which are complex quantities and may have a different density and temperature dependence from the zero frequency transport coefficients. This point should be kept in mind when constructing a phenomenological mode coupling theory for the OCP.[10]

(4) Here we have only considered the long time behavior of the velocity correlation functions for the transport coefficients. We have developed an ordering scheme that enables us to extract the most important contribution in the limit $z \to 0$ and $k \to 0$, where k is the wave number in the macroscopic hydrodynamic equations. In time language, this corresponds to the purely decaying $\sim t^{-3/2}$ long time tails. These tails can also be shown to lead to nonanalyticities in the small-k expansion of the hydrodynamic dispersion relations, in analogy to the neutral fluid case.[18] The oscillating LTT do not contribute to the leading long time decay of the correlation functions, as discussed, nor do they lead to nonanalyticities in the dispersion relations for $z \to 0$ and $k \to 0$.

The dispersion relations for the plasma modes contain, however, finite frequency transport coefficients, evaluated at the plasma frequency, ω_p. It would be of interest to evaluate the leading contribution to the Laplace transform of the Green–Kubo integrands for $z \sim i\omega_p$ and $k \to 0$. To do this, ones needs to reexamine our theory and develop a new ordering scheme. We expect that the oscillating long time tails will give the most important contribution to the frequency-dependent transport coefficients in the region $\omega = iz \sim \omega_p$. Furthermore, the oscillating tails will lead to nonanalyticities in the expansion of these transport coefficients for $\omega \sim \pm\omega_p$, resulting in nonanalytic corrections in the dispersion relations of the plasma modes.

(5) All ring operators $\Delta B_{\alpha\beta}$, for $\alpha\beta = \theta\theta$, θT, Tθ, TT, lead to oscillating long time tails, regardless of whether the first and last collisions

are weak or strong. All these oscillating LTT have the same time dependence, but a different dependence on the plasma parameter, ε_p. For small ε_p the leading contribution is $C^{(1)}_{D_{\theta\theta}}(t)$. Denoting the oscillating tail due to the strong collision ring operator by $\tilde{C}^{(1)}_{D_{\mathrm{TT}}}(t)$, we have

$$C^{(1)}_{D_{\theta\theta}}(t)/\tilde{C}^{(1)}_{D_{\mathrm{TT}}}(t) \sim \frac{1}{\varepsilon_\rho^2}$$

and

$$C^{(1)}_{D_{\theta\mathrm{T}}}(t)/\tilde{C}^{(1)}_{D_{\mathrm{TT}}}(t) \sim \frac{1}{\varepsilon_p}$$

The contribution from the weak-collision ring operator $\Delta B_{\theta\theta}$ could then give important correction to the nonanalyticities in the dispersion relation of the plasma modes discussed in (4) above.

(6) In this paper, we have only considered the kinetic part of the correlation functions that determine the shear viscosity, η, and the heat conductivity, λ. As discussed in Section 1, the potential parts of the time correlation functions for η and λ have long time tails $\sim t^{-3/2}$ whose coefficients are of the same order in the plasma parameter as the coefficients of the LTT of the kinetic part. Therefore, we have *not* given here the complete long time tails of the Green–Kubo integrands for η and λ to lowest order in the plasma parameter. In particular, we have found that two mechanism contribute to the $t^{-3/2}$ long time decay of the kinetic part of the shear current autocorrelation functions: the coupling of two shear modes and that of two plasma modes of opposite sign. The latter contribution is, however, exactly canceled in the full Green–Kubo integrand by a $t^{-3/2}$ tail contained in the potential part of this correlation function. The coupling of two plasma modes can then lead only to an oscillating tail in the Green–Kubo integrand. In the case of the heat current autocorrelation function no such cancellation occurs. There too the potential part contains, however, long time tails that are at least as important as those evaluated here. The long time behavior of the potential part of the Green–Kubo integrands has been evaluated elsewhere using a phenomenological mode coupling theory.[10]

Computer molecular dynamic experiments can measure the kinetic and the potential parts of the time correlation functions separately. The results given here for the purely kinetic contribution have therefore intrinsic interest.

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APPENDIX A

In this Appendix we prove the properties of the vertices $T^R_{s,H}$ and $T^L_{s,H}$ that are needed in the text of the paper.

A1. Proof of $T^L_{s,H} = [T^R_{s,H}]^\dagger$

The adjoint (denoted by a dagger) of an operator $\hat{O}(\mathbf{v}_1, \mathbf{v}_2)$ is defined by

$$\int d\mathbf{v}_1\, d\mathbf{v}_2\, A(\mathbf{v}_1, \mathbf{v}_2)\, \hat{O}(\mathbf{v}_1, \mathbf{v}_2)\, B(\mathbf{v}_1, \mathbf{v}_2)\, \phi(v_1)\, \phi(v_2)$$

$$= \int d\mathbf{v}_1\, d\mathbf{v}_2 [\hat{O}^\dagger(\mathbf{v}_1, \mathbf{v}_2)\, A(\mathbf{v}_1, \mathbf{v}_2)\, B(\mathbf{v}_1, \mathbf{v}_2)\, \phi(v_1)\, \phi(v_2) \qquad \text{(A1)}$$

where A and B are arbitrary functions. Here we will only prove that the above relationship holds for the first two terms in the density expansion of $T^R_{s,H}$ and $T^L_{s,H}$. To lowest order in the density, the right and left vertices are equal and are both given by Eq. (3.3). We then have immediately,

$$[T^{R(0)}_{s,H}]^\dagger = T^{L(0)}_{s,H} \qquad \text{(A2)}$$

To first order in the density, $T^R_{s,H}$ is given by

$$T^{R(1)}_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}_1, z) = \int \frac{d\mathbf{q}}{(2\pi)^3} \int d\mathbf{v}_3\, \theta_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{12}}$$

$$\times \left[\theta_{\mathbf{q}}(\mathbf{v}_2, \mathbf{v}_3) \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{13}} + (-\beta V_q) \right] [\theta_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_3) + \bar{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_3)] \qquad \text{(A3)}$$

and its adjoint is

$$[T^{R(1)}_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}_1, z)]^\dagger$$

$$= \int \frac{d\mathbf{q}}{(2\pi)^3} \int d\mathbf{v}_3 \theta_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_3) \left\{ \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{13}} [\theta_{-\mathbf{q}}(\mathbf{v}_2, \mathbf{v}_3) + \bar{\theta}_{-\mathbf{q}}(\mathbf{v}_2, \mathbf{v}_3)] + (-\beta V_q) \right\}$$

$$\times \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{12}} [\theta_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) + \bar{\theta}_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)] \qquad \text{(A4)}$$

The left vertex $T^{L}_{s,H}$ to first order in the density is given by

$$T^{L(1)}_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}_1, z) = \int \frac{d\mathbf{q}}{(2\pi)^3} \int d\mathbf{v}_3\, \theta_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_3) \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{13}}$$
$$\times \left[\theta_{-\mathbf{q}}(\mathbf{v}_2, \mathbf{v}_3) \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{12}} + (-\beta V_q) \right] [\theta_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2) + \bar{\theta}_{\mathbf{q}-\mathbf{q}_1}(\mathbf{v}_1, \mathbf{v}_2)] \tag{A5}$$

Inserting the identity

$$\frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{13}} = \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{12}} + \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{12}}\, i\mathbf{q}\cdot\mathbf{v}_{23} \frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{13}} \tag{A6}$$

into the part of $T^{L(1)}_{s,H}$ that contains static corrections and using

$$\bar{\theta}_{-\mathbf{q}}(\mathbf{v}_2, \mathbf{v}_3) = -\beta V_q i\mathbf{q}\cdot\mathbf{v}_{23} \tag{A7}$$

we can see immediately that the right-hand sides of Eqs. (A4) and (A5) are indeed identical.

A2. Small-Frequency Limit of $T^{R,L}_{s,H}$ and Proof of Eqs. (4.8)

The objective here is to derive an explicit representation for the z, $\mathbf{q}\to 0$ limit of the vertices $T^{R,L}_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$. This representation will enable us to prove Eqs. (4.8).

We consider the quantity

$$M(z) = n^2 \int d\mathbf{v}_1\, d\mathbf{v}_2\, A(\mathbf{v}_1, \mathbf{v}_2)\, T^{R}_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\, \phi(v_1)\, \phi(v_2)\, F(\mathbf{v}_1) \tag{A8}$$

for z real and $z > 0$, where A and F are arbitrary functions, and evaluate

$$M(0) = \lim_{z\to 0^+} M(z)$$
$$= \lim_{z\to 0^+} n^2 \int d\mathbf{v}_1\, d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3} A(\mathbf{v}_1, \mathbf{v}_2)\, \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) [z - i\mathbf{q}\cdot\mathbf{v}_{12} - \mathscr{V}_{\mathbf{q}}(\mathbf{v}_2)]^{-1}$$
$$\times\, \phi(v_1)\phi(v_2)\, \hat{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\, F(\mathbf{v}_1) \tag{A9}$$

Equation (A9) can be rewritten in a more explicit form as

$$M(0) = \lim_{z\to 0^+} n^2 \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dz'}{2\pi i} \int d\mathbf{v}_1\, d\mathbf{v}_2\, d\mathbf{v}'\, A(\mathbf{v}_1, \mathbf{v}_2) \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\, S_{\mathrm{DH}}(q)$$
$$\times \frac{1}{z - z' - i\mathbf{q}\cdot\mathbf{v}_1} U_{\mathbf{q}z'}(\mathbf{v}_2 \mid \mathbf{v}')\, \phi(v_1)\, \phi(v')\frac{1}{m} \boldsymbol{\varepsilon}_{-\mathbf{q}}\cdot\frac{\partial}{\partial \mathbf{v}'} F(\mathbf{v}') \tag{A10}$$

Here γ defines a contour in the complex z plane, to the right of all the singularities of the integrand and we have required $\mathrm{Re}(z-z')>0$. We have also introduced the Laplace transform of the Green's function of the Vlasov equation, given by

$$U_{\mathbf{q}z}(\mathbf{v}\,|\,\mathbf{v}')=\int_0^\infty dt\, e^{-zt}\, U_{\mathbf{q}}(\mathbf{v}, t\,|\,\mathbf{v}', 0) \tag{A11a}$$

for $\mathrm{Re}z>0$, where the Green's function $U_{\mathbf{q}}(\mathbf{v}, t\,|\,\mathbf{v}', 0)$ is the solution of the equation

$$\left[\frac{\partial}{\partial t}+i\mathbf{q}\cdot\mathbf{v}-\mathscr{V}_{\mathbf{q}}(\mathbf{v})\right] U_{\mathbf{q}}(\mathbf{v}, t\,|\,\mathbf{v}', 0)=\delta(t)\,\delta(\mathbf{v}-\mathbf{v}') \tag{A11b}$$

The Laplace-transformed Green's function is explicitly given by

$$U_{\mathbf{q}z}(\mathbf{v}\,|\,\mathbf{v}')=\frac{1}{z+i\mathbf{q}\cdot\mathbf{v}}\left[\delta(\mathbf{v}-\mathbf{v}')+\frac{1}{m}\,\boldsymbol{\varepsilon}_{-\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{v}}\,n\phi(v)\,\frac{1}{D(\mathbf{q}, z)}\,\frac{1}{z+i\mathbf{q}\cdot\mathbf{v}'}\right] \tag{A11c}$$

where $D(\mathbf{q}, z)$ is the plasma dispersion function defined as

$$D(\mathbf{q}, z)=1-n\int d\mathbf{v}\,\frac{1}{z+i\mathbf{q}\cdot\mathbf{v}}\,\frac{1}{m}\,\boldsymbol{\varepsilon}_{\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{v}}\,\phi(v) \tag{A11d}$$

for $\mathrm{Re}z>0$.

Inserting Eq. (A11c) into Eq. (A10), we obtain

$$\begin{aligned} M(0)=\lim_{z\to 0^+} n^2 \int_{\gamma-i\infty}^{\gamma=i\infty} \frac{dz'}{2\pi i}\int d\mathbf{v}_1\, d\mathbf{v}_2\, A(\mathbf{v}_1, \mathbf{v}_2)\int \frac{d\mathbf{q}}{(2\pi)^3}\,\theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)\, S_{\mathrm{DH}}(q) \\ \times\frac{1}{z-z'-i\mathbf{q}\cdot\mathbf{v}_1}\,\frac{1}{z'+i\mathbf{q}\cdot\mathbf{v}_2}\left[\phi(v_2)+\frac{1}{m}\,\boldsymbol{\varepsilon}_{\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{v}_2}\,n\phi(v_2)\,\frac{\mathscr{A}(\mathbf{q}, z')}{D(\mathbf{q}, z')}\right] \\ \times\,\phi(v_1)\,\frac{1}{m}\,\boldsymbol{\varepsilon}_{-\mathbf{q}}\cdot\frac{\partial}{\partial\mathbf{v}_1}\,F(\mathbf{v}_1) \end{aligned} \tag{A12}$$

where, for $\mathrm{Re}z'>0$,

$$\mathscr{A}(\mathbf{q}, z')=n\int d\mathbf{v}\,\frac{\phi(v)}{z'+i\mathbf{q}\cdot\mathbf{v}} \tag{A13}$$

The z' integral in Eq. (A12) can be performed by closing the contour to the left. Changing $\mathbf{q}$ into $-\mathbf{q}$ and using

$$D(-\mathbf{q}, z+i\mathbf{q}\cdot\mathbf{v})=[D(\mathbf{q}, z^*-i\mathbf{q}\cdot\mathbf{v})]^* \tag{A14}$$

together with a similar identity for $\mathcal{A}(\mathbf{q}, z)$, we obtain

$$M(0) = \lim_{z \to 0^+} n^2 \int d\mathbf{v}_1 \, d\mathbf{v}_2 \, A(\mathbf{v}_1, \mathbf{v}_2) \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \, S_{\mathrm{DH}}(q) \, \phi(v_1)$$
$$\times \left\{ \pi\delta(\mathbf{q}\cdot\mathbf{v}_{12}) \, \phi(v_2) - \frac{1}{m} V_q \mathbf{q} \cdot \frac{\partial}{\partial \mathbf{v}_2} n\phi(v_2) \right.$$
$$\left. \times \, \mathrm{Im} \left[\frac{1}{z - i\mathbf{q}\cdot\mathbf{v}_{12}} \frac{\mathcal{A}(\mathbf{q}, z - i\mathbf{q}\cdot\mathbf{v}_1)}{D(\mathbf{q}, z - i\mathbf{q}\cdot\mathbf{v}_1)} \right] \right\} \frac{1}{m} \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} F(\mathbf{v}_1) \qquad \text{(A15)}$$

The evaluation of the imaginary part in Eq. (A.15) is straightforward, but lengthy. The result is

$$M(0) = \lim_{z \to 0^+} n^2 \int d\mathbf{v}_1 \, d\mathbf{v}_2 \, A(\mathbf{v}_1, \mathbf{v}_2) \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \frac{\pi S_{\mathrm{DH}}(q)}{|D(\mathbf{q}, -i\mathbf{q}\cdot\mathbf{v}_1)|^2}$$
$$\times \, \phi(v_1) \, \phi(v_2) \left\{ \delta(\mathbf{q}\cdot\mathbf{v}_{12})(1 + \beta V_q) \right.$$
$$\times \left[1 + n\beta V_q P \int d\mathbf{v}_3 \frac{1}{\mathbf{q}\cdot\mathbf{v}_{31}} \mathbf{q}\cdot\mathbf{v}_3 \, \phi(v_3) \right]$$
$$\left. + \, n\beta V_q \mathbf{q}\cdot\mathbf{v}_2 P \frac{1}{\mathbf{q}\cdot\mathbf{v}_{12}} \int d\mathbf{v}_3 \, \delta(\mathbf{q}\cdot\mathbf{v}_{31}) \, \phi(v_3) \right\} \frac{1}{m} \boldsymbol{\varepsilon}_{-\mathbf{q}} \cdot \frac{\partial}{\partial \mathbf{v}_1} F(v_1) \qquad \text{(A16)}$$

Here P denotes the principal part of the integral. From Eq. (A16) we can immemdiately identify the $\mathbf{q}$, $z \to 0$ limit of the right vertex $T^R_{s,H}$. It is given by

$$T^R_{s,0}(\mathbf{v}_1, \mathbf{v}_2) = \lim_{z \to 0^+} \lim_{\mathbf{q} \to 0} T^R_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)$$
$$= \int \frac{d\mathbf{q}}{(2\pi)^3} \theta_{\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) \frac{\pi S_{\mathrm{DH}}(q)}{|D(\mathbf{q}, -i\mathbf{q}\cdot\mathbf{v}_1)|^2} \left\{ \delta(\mathbf{q}\cdot\mathbf{v}_{12})(1 + n\beta V_q) \right.$$
$$\times \left[1 + n\beta V_q P \int \frac{1}{\mathbf{q}\cdot\mathbf{v}_{31}} \mathbf{q}\cdot\mathbf{v}_3 \phi(v_3) \right]$$
$$\left. + \, n\beta V_q \mathbf{q}\cdot\mathbf{v}_2 P \frac{1}{\mathbf{q}\cdot\mathbf{v}_{12}} \int d\mathbf{v}_3 \, \delta(\mathbf{q}\cdot\mathbf{v}_{31}) \, \phi(v_3) \right\}$$
$$\times \, [\theta_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2) + \bar{\theta}_{-\mathbf{q}}(\mathbf{v}_1, \mathbf{v}_2)] \qquad \text{(A17)}$$

Using Eq. (A17) it is easy to see that $T^R_{s,0}$ is indeed the kernel of the Markovian "self" BGL operator, i.e.,

$$\begin{aligned}\tilde{\Lambda}^0_s(\mathbf{v}_1, 0)\,\phi(v_1)\,F(\mathbf{v}_1) &= n \int d\mathbf{v}_2\, T^R_{s,0}(\mathbf{v}_1, \mathbf{v}_2)\,\phi(v_2)\,\phi(v_1)\,F(\mathbf{v}_1)\\ &= -\frac{n}{m^2} \int d\mathbf{v}_2 \int \frac{d\mathbf{q}}{(2\pi)^3}\,\mathbf{q}\cdot\frac{\partial}{\partial \mathbf{v}_1}\,\frac{\pi V_q^2}{|D(\mathbf{q}, -i\mathbf{q}\cdot\mathbf{v}_1)|^2}\\ &\quad\times \delta(\mathbf{q}\cdot\mathbf{v}_{12})\,\phi(v_1)\,\phi(v_2)\,\mathbf{q}\cdot\frac{\partial}{\partial \mathbf{v}_1}\,F(\mathbf{v}_1) \qquad \text{(A18)}\end{aligned}$$

where the explicit representation of $\tilde{\Lambda}^0_s(\mathbf{v}_1, 0)$ given in the second equally of Eq. (A18) can be derived as indicated in I. The Markoffian operator $T^R_{s,0}$ has also the property

$$\int d\mathbf{v}_2\, T^R_{s,0}(\mathbf{v}_1, \mathbf{v}_2)\,\phi(v_1)\,\phi(v_2)[\psi_j(\mathbf{v}_1) + \psi_j(\mathbf{v}_2)] = 0 \qquad \text{(A19)}$$

where $\psi_j(\mathbf{v}) = \{1, \mathbf{v}, \mathbf{v}^2\}$.

Making use of the above properties, Eq. (4.8a) is now easily proved. Using that for small q, $\Theta^R_s(\mathbf{v}, \mathbf{q}) = 1 + O(\mathbf{q})$, we obtain

$$\begin{aligned}&\lim_{z\to 0^+}\lim_{q\to 0} \langle v_{1x}|\,[\tilde{\Lambda}^0_s(\mathbf{v}_1, 0)^{-1} T^R_{s,H}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{q}, z)\,|\Theta^R_s(\mathbf{v}_1, -\mathbf{q})\,\Theta^R_\alpha(\mathbf{v}_2, \mathbf{q})\rangle_\phi\\ &\quad= -n^{-1}\langle v_{1x}|\,\Theta^R_{\alpha,0}(\mathbf{v}_1, \mathbf{q})\rangle_\phi \qquad \text{(A20)}\end{aligned}$$

where we have used that $\Theta^R_{\alpha,0}(\mathbf{v}, \mathbf{q})$ is a linear combination of the five collision invariants $\psi_j(\mathbf{v})$.

Finally, the analogous properties for the left vertex $T^L_{s,H}$ can be proved in the same way.

REFERENCES

1. H. Gould and G. F. Mazenko, *Phys. Rev. A* **15**:1274 (1977).
2. M. Baus and J. Wallenborn, *J. Stat. Phys.* **16**:91 (1977).
3. J. A. Krommes and C. Oberman, *J. Plasma Phys.* **16**:193 (1976).
4. P. Vieillefosse and J. P. Hansen, *Phys. A* **12**:1106 (1975) and references therein; see also M. Baus and J. P. Hansen, *Phys. Rep.* **59**:1–94 (1980).
5. B. Bernu and J. P. Hansen, *Phys. Rev. Lett.* **48**:1375 (1982).
6. M. Baus and J. P. Hansen, *Phys. Rep.* **59**:52 (1980).
7. Y. Pomeau and R. Resibois, *Phys. Rep.* **19**:63 (1975).
8. M. C. Marchetti, T. R. Kirkpatrick, J. R. Dorfman, and E. G. D. Cohen, *J. Stat. Phys.* **41**:37 (1985).
9. T. Gaskell, *J. Phys. C* **15**:1601 (1982).
10. M. C. Marchetti and T. R. Kirkpatrick, *Phys. Rev. A* **32** (1985).

11. J. R. Dorfman and E. G. D. Cohen, *Phys. Rev. A* **6**:776 (1972); *Phys. Rev. A* **12**:292 (1975).
12. L. P. Kadanoff and J. Swift, *Phys. Rev.* **166**:89 (1968).
13. G. E. Uhlenbeck and G. W. Ford, in *Studies in Statistical Mechanics I*, J. de Boer and G. E. Uhlenbeck, eds. (North-Holland, Amsterdam, 1961).
14. R. Zwanzig, *Phys. Rev.* **129**:486 (1963).
15. M. H. Ernst, . H. Hauge, and J. M. J. van Leeuwen, *Phys. Rev. A* **4**:2055 (1971).
16. M. Baus, in *Strongly Coupled Plasma*, G. Kalman and P. Carini, eds. (Plenum Press, New York, 1978), p. 279.
17. T. R. Kirkpatrick and E. G. D. Cohen, *J. Stat. Phys.* **33**:639 (1983).
18. M. H. Ernst and J. R. Dorfman, *J. Stat. Phys.* **12**:311 (1975).
19. R. L. Guernsey, in *Lectures in Theoretical Physics-Kinetic Theory*, Vol. IXC, W. E. Brittin, ed. (Gordon and Breach, New York, 1967), p. 147.